The study of the genesis of novel mathematical and mechanical theories provides an inspiration for future original research

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Abstract

In this introductory chapter we present the motivations that prompted the authors and editors to work on this volume. The *fil rouge* followed in the discussion presented below is based on the following consideration: the study of the genesis of mathematical and mechanical theories does not have a merely philological purpose, but can influence and even inspire the development of new original ideas. We present some examples that clarify our thesis: the development of the model of planetary motion and a historical-critical study of the development of Continuum Mechanics. Clearly understanding the main errors and misunderstandings that other than pure research logics have introduced in the scientific discussion is the only way to learn a serious and rigorous approach to Science. An idea that has guided us in the development of this chapter is that fragmentation of culture brings to inability to deal with complexity. The only way to study and, above all, understand the complexity of our world is through a unified vision of knowledge.

1 Introduction

This Introductory Chapter is conceived in order to make explicit the motivations that led the authors and the editors to work on this volume. The reader will find additional arguments and considerations on some of the epistemological and methodological questions discussed here in the Chapter referred to in [1]: we will however try to present self-consistent reasonings, so that one is not expected to complement this chapter with other readings, if she/he does not wish. The question we want to face is simply stated as follows: Can we find a meta-theory teaching us to formulate a set of specific theories each of them being suitable to describe a well-precise set of phenomena? Unfortunately, it seems that to this question there are not fully satisfactory answers yet. There is not, in facts, any kind of *algorithm* following which one can construct a reasonably efficient theory: whatever it may be said by the supporters of *Data Science* it is not possible, in this moment of the scientific development, to replace the creativity act of a scholar in formulating a model with any kind of *Big Data* algorithm.

We do not want to say that such a possibility is precluded to humankind: after having invented robots that relieved us from the greatest part of manual work, it is possible, if not likely, that in future we will be relieved by Artificial Intelligence from the greatest part, or maybe from all, the intellectual work. What has to be clear is that, notwithstanding the trends and pretensions of many, the ambitious program of replacing human mind in its formulation of mathematical models is by far out of the reach of present times science. We will not dare to give some timeline indication concerning the occurrence of such a gigantic innovation, as the second author did remember very well when a famous scientist, who was his professor in electronics, announced that it was not conceivable the construction of a computer which could beat a human champion in a chess game. It was not earlier than 1983, and *Deep Blue* in 1996 did manage in the endeavor. We want to underline that we are, however, confident that such a progress will occur, and, that it will revolutionize our life, possibly our species biology and, surely, it will open a new era in *Natural History*. However exactly as Eugenics did not represent any true advancement of science (nothing even barely comparable with modern molecular genetics, whose successes seem to be limitless), present time Data Science seems to be a fashion that is simply exploited by some scholars who are trying to get more academic (and maybe economical) power. The situation, as realistically presents itself in the current historical moment, is really clear: one can teach to young generations how to formulate a scientific model in only one way, that is by showing them how successful scientific models were at first formulated. This aim motivates the entire content of this work.

2 The process of knowledge transmission: a sociological problem that needs to be studied by using the scientific method

The present work has been produced by the collaboration of scholars whose competences are relatively varied. However all of them never accepted a deleterious concept that is at the basis of modern organization of scientific research: that which lead to fragmentation of knowledge into hyper-specialized sub disciplines rigidly divided by sharp boundaries.

2.1 Fragmentation of culture brings to inability to deal with complexity

Albeit nearly all authors and editors can be defined to be (Applied) Mathematicians, Physicists (or Mechanicians) or Engineering Scientists, all of them were exposed to humanistic culture and greatly value multidisciplinar studies. They all agreed that it is really dangerous, and surely pitiful, that, in the modern more fashionable attitudes of academic milieux, the fundamental unity of human knowledge is being unrecognized as a founding and strong feature of scholarly activity. As a consequence, quickly and systematically, Western Culture has experienced a decay of the capacity of addressing complex problems with a unitary vision of all their various facets.

In the short span of few scholars' generations there was a dramatic change in the perception of the role of a scholar: in present times the fields of expertise are being more and more carefully delimited by boundaries that are more and more difficult to trespass. Therefore one may wonder if, in present days, a scholar like Johan Ludvig Heiberg (1854-1928) would have been produced by the current academic system. Heiberg not only mastered fully the Ancient Greek language, in its main versions, including Doric Greek, together with Latin. What makes his personality really unique is that he has systematically shown to master some not easy parts of mathematics, as he could perfectly understand very deep texts by Archimedes, those contained in the famous Palimpsest. Postponing to the subsequent chapters of this work some relevant and more detailed discussion about this text, we start recalling here that it is a parchment codex palimpsest, containing, after having scratched the first written text, some prayers. How it could be possible that a scholar did manage to abrade a text by the greatest recognized scientist who ever wrote in Greek language has to be studied carefully, and is one of the problems that we signal for future scientific investigations. In facts the original text is a Byzantine Greek copy of a compilation of works by many authors among whom Archimedes seems to be the most prominent. This original text contained two previously unknown very important texts by Archimedes i.e. the *Stomachion* and the *Method of Mechanical Theorems* (which is shortly referred to as *The Method*). Moreover it contains also the only surviving original Greek manuscript of the celebrated Archimedean work on *On Floating Bodies.* Heiberg, following the tradition of Western Culture, did translate the Greek Archimedean text into Latin, whose role of *lingua franca* has been recognized nearly universally. Unfortunately after the 1906 Heiberg's discovery and the subsequent publication of his *Archimedis Opera Omnia* (i.e. The complete works of Archimedes), and after a short period in which French seemed to have replaced Latin, English established itself as the modern, possibly even more universal, *lingua franca.*

As a consequence of this sudden change, a text written in Latin (but unfortunately also if it was written in any living language different from English) became not readable by nearly every modern scholar (sometimes even by some professor of Latin Language!). As it always happened when a change of the used language in science occurred, there is a very likely phenomenon that systematically occurs: a large part of the knowledge accumulated in the old language is forgotten and lost. Another part of this knowledge resurfaces in the new dominant language (the part of Archimedean results that resurfaced thanks to Tartaglia gives an example of such a phenomenon, see the following chapters) and is *rediscovered* several times. This rediscovers occurs in several difference space locations, in different times (many anachronisms may be explained in this way) and also in different languages. Useless to say, this process of systematic rediscovery slows a lot the advancement of science and is really detrimental, as it systematically causes regressions in technology.

An example of the rediscovery of a body of knowledge lost because of linguistic barriers that we will examine concerns the works by Gabrio Piola (see [2–6]). Piola's work were nearly completely ignored for a long period and were recovered because of a series of fortuitous events. In facts Piola's works were written in Italian and because of the wrong choice of the used language their diffusion was strongly limited. In this work we will prove that there are, also in mechanical science, very interesting ideas that were originally written in different languages than English.

The aim of the present work is to prove that, differently from what has been too often conjectured, scientific knowledge transmission is not a simple process: the vision of science as a continuous and endless progress from less advanced to more advanced stages has been falsified even too many times in the history of science. Surely there are the problems related to linguistic barriers, when the *lingua franca* changes because of one of the many possible social reasons. Many scientific ideas were lost in translation! However there are also some psychological and barely *survival* mechanisms that causes erasures, loss and deformation of the scientific knowledge in its transmission process. These mechanisms play a crucial role in the advancement of science, whatever may be believed by some *right-thinking* scholars. These scholars believe that one has to avoid the consideration, when studying knowledge transmission processes, of the social phenomena related to jealousy, revenge, inflated self-esteem, bare ignorance, arrogance, need of earning from academic positions, every form of nepotism and use of scientific knowledge for getting any form of power. Many are embarrassed when the existence of these social and psychological mechanism are evoked and when one expresses the opinion that they may play a crucial role in the rise of any form of Dark Ages. In facts their consideration is considered not politically correct and trying to take into account their influence in history of science a form of mental disorder of the kind of paranoia. Instead, exactly as Alfred Kinsey has scientifically shown how important is sexuality in human life and in the psychopathology of humankind, we believe that the social forces that are shaping human psychology are of great relevance in the mechanisms that produce scientific research. Such an obvious statement, as a similarly obvious consequence, implies that it is possible that a deep scientific theory, a useful body of knowledge or an effective mathematical model may be erased, lost, or, in the best case, forgotten for a while in a scientific group, simply because of a series of socio-psychological reasons which are completely unrelated to their absolute scientific merit. Aforementioned *right-thinking* scholars will claim that science is *objective* and that even considering the possibility of any influence of the dark side of human mind on its development is harmful for humankind. This reasoning may be considered equivalent to believing that one can defeat an epidemics simply ignoring its existence: an action whose consequences are well-known. The story of the struggles of Tartaglia (1499-1557) to persuade all his contemporaries that he could understand and translate the Archimedean works, as reconstructed objectively by Heiberg, gives us a paradigmatic and incontrovertible evidence that our thesis is very well-grounded.

2.2 Why to try to establish how and when a scientific theory was first formulated? Difficulties in this endeavour

In the discussion that we will develop in our work we will focus on at least two important aspects of the considered question. The first aspect concerns the importance of study of the true origins of the scientific theories. One may argue that the value of a scientific theory resides in its predictive capacity, and that it is enough to supply a whatsoever rigorous and precise formulation for it. If one accepts this point of view then, when a theory is formulated in a way or in another equivalent way then she/he can choose the preferred way based on any reasonable and useful criterion. Our point of view is, instead, that if one wants to learn how to formulate a completely new theory, a theory that was never formulated before, she/he has to learn, in absence of the meta-theory invoked and dreamed before, how the successful theories have been formulated first, and how they were subsequently developed. To see how old and established theories were born may be of use in the process of inventing a completely new one. In facts, we do not have, presently, any way to supply to younger generations any other well-working method for teaching them how to build theories that are efficiently capable to give the correct predictions for both observed and not-yet-observed phenomena.

As a consequence we are obliged to follow the educational methods of those ancient *Renaissance Maestri*, who trained their pupils to sculpture or painting simply by showing them as the *Maestro* was painting or sculpting. Unfortunately there are very few great *Maestri* alive in a certain historical moments and, moreover, their workshops are already full of pupils. Therefore one has to show to those young scholars, who aspire to invent something original, how the available theories, relevant in the chosen disciplines, were first conceived and developed: in this way we hope that the lesson given by great scholars example will guide new generations. For this reason a presentation of available theories must follow, as carefully as possible, the original invention process that led their inventors to get them[1].

The second aspect, on which our analysis will particularly focus, concerns the process of transmission of knowledge from competent scholars to competent scholars via intermediate scholars who are not so competent. Albeit the transmission of science is based on written texts, the role of the scholars participating to the editing of the texts and using them as textbooks for their young pupils cannot be neglected. When the books were handwritten, their relatively enormous economical value introduced a further selection filter in knowledge transmission: the economical costs imposed a selection of what could be copied and what deserved oblivion. In this choice the Archimedean Palimpsest was sacrificed for a book of prayers against diseases, a subject that seemed more "practical" than abstract mathematics. A scholar choosing what kind of text-

[1]The second author is greatly indebted to Prof. Roberto Stroffolini (Università di Napoli Federico II) for having shown him how such a teaching method has to be pursued.

book deserves to be transmitted plays a relevant role also in the era of printed books: many books are not reprinted and remain in fewer and fewer exemplars in the storages of libraries, virtually disappearing from the attention of younger generations. In our (unfortunate) époque of citations metrics another method has been conceived to condemn to oblivion a certain textbook, authors or theory: it is enough to forget to cite them, and soon nobody will find these works in the *mare magnum* of modern literature, which is literally overflown with too many ripetitive and not original papers and textbooks.

Finally, an influential compiler of a textbook, having many students may influence many of them with his biased choices. In the milieux of mechanical sciences there are many textbooks that were very successful in transmitting the correct ideas to clever students, albeit it is clear that their compilers did not understand very much the scientific results that they had carefully copied from reliable sources. There are, also, examples of textbooks that deformed the true intent of their sources, imposing to too many younger scholars wrong points of view or making for them every original research extremely difficult. We will fully describe, under the guidance of the authoritative Heiberg's analysis, how Tartaglia did manage to have a relevant role in the *translation* in the *language* used by Western Science of some of the most relevant works by Archimedes. Albeit this may seem rather simple (and most likely also very useful), we will not try to establish any relationship between the publishing (and survival) strategy chosen by Tartaglia and that chosen by (too) many more modern scientists. In facts, the need of getting a salary seems to allow for any kind of deplorable choice, while Tartaglia features a "representative" scholar, belonging to a specific kind. This kind of scholar is observed nearly ubiquitously in history of science: one can find examples of it in any social group, language, scientific discipline, historical period, geographical location and economical and political organization.

Instead of looking for specific examples of such kind of scholar, we will try to phenomenologically describe their behavior, the effects of their existence on science transmission and on its accumulation and loss. We will try to apply the scientific method in our phenomenological description and in our first efforts of looking for a model of it. The phenomenology can be shortly resumed as follows: in the competition that they need to accept in order to have recognized their own scientific capacities, many scholars systematically want to ignore any signal indicating that they are not original enough to deserve an academic position. They badly need the sinecura that they believe to be associated to it, and therefore try to prove, in any possible way, that they do deserve highly ranked positions. If they meet somebody indicating how weak their scientific skills are, then they may react in two different ways: i) they start believing that there is a conspiracy against them or ii) albeit they may understand that the criticism against them is well-founded, they manage to persuade themselves that since there are so many incompetent scholars, then their own exclusion from academia is not moral. These scholars, either if they are conscious of their weaknesses or if they sincerely believe to be clever enough for their ambitions, try to make their best to persuade all other scholars that they can be considered original thinkers.

Sometimes, exactly as it was done by Tartaglia, these self-proclaimed scientists *reformulate*, *make rigorous*, *translate*, *clarify* or *make more precise* works that they have found in the literature. Exactly as Tartaglia included in the title of one of his presumed translations the following statement: "here I make clear what it was not possible to understand in Archimedes works", his epigones manage to declare that they "clarified" the previously "obscure" theories, while in fact they are completely misunderstanding them.

2.3 To unveil the real contribution of Tartaglia (and his Encyclopaedic or polymath epigones) to science is not easy

The capacity of some scholars in avoiding any discussion about the merit of their scientific contributions is legendary. They manage to bend even mathematical argument to their aims, making any discussion about what they claim to have discovered completely useless. One has to avoid any effort in trying to prove that a single specific scholar is not producing any original contribution or any original view in presenting already known results. Instead it is very useful to describe from a general point of view the kind of effect that the existence of the aforementioned type of scholars has on science transmission and development. If this phenomenology is understood then, most likely, some countermeasures can be acted to limit the unavoidable impact of such scholars on the destinies of science.

Albeit this information seems to have been somehow forgotten, Heiberg happened to discover, while reordering and preparing for his edition the whole available texts by Archimedes, that, in reality, the only merit one can attribute to Tartaglia, for what concerns the appreciation of Archimedes work, is purely propagandistic. Tartaglia contributed to revive the interest in Archimedes. Heiberg, while prefacing his Complete Works by Archimedes, gathered all necessary evidence to prove that Tartaglia's capacity in writing in a correct Latin was rather scarce. One can deduce therefore that he could never have the possibility to translate, from the Doric Greek used by Archimedes into Latin, a complex text of advanced mathematics.

Heiberg argument seems to us very detailed, serious and careful: unfortunately this argument was buried in the Prolegomena of the famous Archimedes Edition. We could say it was buried since this Prolegomena (as well as the whole translation of Greek text) was written into Latin. While there are many valuable translations into English of Heiberg's Latin text, the Prolegomena, to our knowledge, were never translated into any modern language. Therefore we were motivated to translate in this work aforementioned Prolegomena and to add our own comments to it, in order to highlight those aspects of the phenomenology of knowledge to which we are particularly interested. The sociological and cultural phenomena that are surfacing from this reading deserve, in our opinion, a great attention.

Their importance cannot, indeed, be underestimated: if one wants to describe carefully the process of birth of a novel theory she/he must establish exactly when, how and in which formulation, it was first conceived. This description is essential for pedagogical aims: younger generations of scientists must learn how to formulate novel theories by looking at the invention process of the most successful ones. The phenomenon of science transmission is rather complex and manifold: one can find many of its aspects that are of great relevance. One that plays an important role concerns the systematizing and paradigmatic role of Encyclopaedias and Encyclopaedic compilations. Because of their true nature, they gather many important aspects of knowledge into a well-organized and unitary way, by using a common formalism and vision. Moreover they give a synthetic account of all human knowledge, in the most ambitious projects, or for a specific group of disciplines, in other cases. Encyclopaedias supply a precious support for subsequent generations of scholars, as they supply a global understanding of the state of the art, in a given group of scholars, place and époque. By sacrificing some technical details, they resume large bodies of knowledge in an agile presentation and indicate where the interested scholar can find the details that she/he may need. However the existence of Encyclopaedic summaries makes more difficult to understand if a certain scholar did really master her/his discipline, or if she/he did simply adsorb superficially one of the available Compendia.

Our attention has been attracted, in this context, by the 1913 Hellinger's Entry of German Encyclopaedia of Mathematics whose aim was to give an overview of then current state of the art in Continuum Mechanics and list some research perspectives that seemed promising to the author. This text has not been translated into English until recently (see [7–9]) and proves that, in facts, Continuum Mechanics has been "frozen" because of the establishment of English as the novel *lingua franca*, and by the incapacity of the community of experts in Mechanics to read French, Italian or German.

The summary and the analysis presented by Hellinger is really clear and far reaching. Moreover the research perspectives, read by somebody in 2021, seem to be visionary: only recently some of them are being developed. It is remarkable that Hellinger could forecast the main directions of future development of Continuum Mechanics with such a large anticipation. The question therefore is: why Hellinger's work has been removed by the list of the most used sources of XX century by the great majority of scholars in Mechanics? A partial answer is that it was written in German. Moreover the author was Jewish and, unfortunately, this did not help the diffusion of his work in the milieu of German speaking Mechanicians, at least until the end of Second World War.

Such an erasure resulted in a great damage to the advancement of Mechanical Sciences. The loss of the consciousness of Hellinger's analysis in the German speaking Mechanics community had rather singular effects. Indeed, while many authors showed to be aware of the results presented in his work, the information about the fundamental fact that these results were, for the first time, obtained by using variational principles was lost. Therefore, exactly as it was done by Tartaglia, the secondary sources from Hellinger (some of them emigrated in the USA, together with their authors) presented some reworks of Hellinger's Compendium in such a way that it was impossible to get from them any hint about the heuristic method used for finding the presented results. These Compendia were presented as if they were a completely original contribution of their authors, who seemed to have had an *out of the blue* inspiration. This feeling impresses on the readers the false belief that science is an *epic* endeavor where few, particularly gifted scientists, *wake up one morning* and without any apparent cause, simply because they are geniuses, manage to invent a novel theory. In facts any theory is the result of a choral work of generations of scholars: what is found in some modern textbooks in Continuum Mechanics is the elaboration, hiding the variational procedure first used for finding them, of the contribution to the discipline given by many scholars, starting from Lagrange [10–12], Piola [2–4, 13, 14], the Cosserat brothers [15–19], and continuing with Sedov [20–22], Toupin [23] and Mindlin [24], among many others [25–29]. The Entry by Hellinger represents a deep scientific contribution to Mechanics, as it originally reorganizes, with the rigor of a gifted mathematician, all results available up to 1913. It could have given an impressive impulse to the development of 20th century Continuum Mechanics if only it had been understood by the scientific community.

It has to be said that there is a possible misuse of the Encyclopaedic Entries, and this misuse concerned also that by Hellinger: indeed, the results presented in this kind of Compendia may be *adsorbed* and *reworked* by Tartaglia's epigones, who will present them from different, and sometimes twisted, points of view. Moreover the existence of Encyclopaedic Entry make possible the existence of so called-polymath scholars: these scholars, who probably have the access to Encyclopaedia Entry, are claimed to have a universal knowledge. Instead, most probably, they simply had access to a, very often lost, Encyclopaedic Entry. In particular Hellinger's work is surely the starting point of the reworking of Continuum Mechanics as presented by those scholars who do refuse Variational Postulation. Knowing in advance the correct results it is easy to deduce them by a series of ad hoc postulates, claiming that they are *induced* by experimental evidence. We will more diffusely present this point in the following sections of this Chapter. Of course this misuse was not intended by Helinger when he conceived his Entry. Unfortunately, until very recently, as a direct source this Entry was completely ignored. We could find a few fugitive mentions of it, where it has been rather harshly criticized. What we have just described is another of the sociological phenomena that must be studied and understood. Understanding it will have an important consequence: thanks to the obtained insight one can find operative methods for organizing the recruitment of academic bodies in a more efficient way.

2.4 Archimedes: "The Method of Mechanical Theorems" is an authoritative source confirming our thesis

To our knowledge, Archimedes is the first known scientist who described explicitly a heuristic way for finding novel theories, theorems and mathematical models. Archimedes' mastering of the concept of "model" of physical reality by using mathematical deductive theories has pushed us to conjecture that his epistemological vision may be considered, in essence, to be that of a falsificationist.

This statement may need further deepening: here we consider sufficient to quote, once more, and to give some further few comments, what Archimedes wrote at the beginning of his “The Method of Mechanical Theorems, for Eratosthenes”. The Archimedean text was written in Doric Greek, and it is a difficult issue to decide which English translation transmits more faithfully the original ideas and spirits. It seems that the scholarly work of those who are capable to understand mathematics, physics, model theory and Doric Greek is very useful also nowadays. The English text which we are going to reproduce here is, in facts, the final results of many transformations: the Greek text found by Heiberg was translated by Heiberg himself into (Modern) Latin (in his celebrated Edition of Archimedes' Works). Heiberg's Latin text was then translated into Dutch by E.J. Dijksterhuis in 1938 and then into English by C. Dikshoorn in 1956 (see [30]). Notwithstanding this subsequent translation we believe to see more clearly the ideas of Archimedes in the presented text than, by analogy, what we can read in the translation presented in [7–9] for the Hellinger's ideas. There are also some hints about the way in which Hellenistic culture organized science in this text. In facts Archimedes starts his “cover letter” by recognizing to Eratosthenes a scholarly preeminence but only as a “manager of scientific research” and as “editor-in-chief” of the publications and manuscripts produced by the library of Alexandria: “Since, as I said, I know that you are diligent, an excellent teacher of philosophy, and greatly interested in any mathematical investigations that may come your way, I thought it might be appropriate to write down and set forth for you in this same book a certain special method, by means of which you will be enabled to recognize certain mathematical questions with the aid of mechanics.”

This preamble may be interpreted as a kind of *captatio benevolentiae.* Now, from all sources we know how great was the fame that Archimedes enjoyed also during his life. Why did he need to be so careful in sending his paper to Eratosthenes? One can conjecture that also in Hellenistic scientific milieux it was possible to observe a phenomenon that to a much larger extent has been developed later: the diffusion of culture happens to be controlled by few powerful scholars, whose decision can greatly influence the destiny of any scientific work, including those written by outstanding persons, as Archimedes was already recognized to be. The existence of “well-established” scientific personalities who had the power to control what can be published or what must be bound to oblivion seems to be therefore attested already at the époque of the library of Alexandria, and seems to be an unavoidable side effect of any form of organization of Big Science.

Eratosthenes of Cyrene (about 276 BC – about 195/194 BC) was probably one of the most influential personality of Hellensitic science. Obviously, having the control and full access to the biggest source of scientific knowledge of antiquity, he is often described as a polymath. It is interesting to remark here that the etymology of the word “polymath” goes back to ancient Greek. The Greek word πολυμαθής can be translated as follows: "[somebody] having learned much". The translation that has been more often used in Latin is: *homo universalis.* i.e., "universal man". We believe that too often polymaths are simply scholars who managed to better reorganize the results found by other, more original, scientists. Very often the compiler of Compendia or Encyclopaedia Entries are this kind of erudite polymath. The most skilled among polymaths, however, are very precious: they allow for the diffusion of specialist theories among a wider set of scholars: we believe that Hellinger, being an original mathematician himself, when accepting to write an Entry about Continuum Mechanics did cleverly master the subject and then could give the best indications about its future paths of development.

Eratosthenes' interests apparently spanned mathematics, poetry, geography, astronomy and music theory. In fact, most likely he was an erudite who managed to persuade the Pharaoh Ptolemy III Euergetes to nominate him as a "chief-librarian" at the Library of Alexandria in the year 245 BC. One has to consider that the choice was really appropriate: as head of such an institution one needs indeed a true and gifted polymath. He was the leader of the group of scientists and technicians that founded scientific geography and he is best known for having directed the group of scholars that obtained a careful calculation of the circumference of the Earth and the tilt of the Earth's axis. He introduced the first global planar projection of the world, by using parallels and meridians. Most likely he has also calculated the distance from the Earth to the Sun and understood the need of the leap day for a precise Calendar. In number theory, the sieve of Eratosthenes, an efficient algorithm for calculating prime numbers is attributed to him. In the entry of the Suda[2] concerning Eratosthenes it is reported that his critics called him Beta (that is: the "second", as beta is the second letter of the Greek alphabet). This scornful attribute had been chosen to underline that he was the second biggest expert in all his domains of competence. On the other hand, without denying this circumstance and even confirming it, his supporters called him *Pentathlos* after the Olympian Athletes competing in the pentathlon, i.e. athletes being "well-rounded" in five different sports. Eratosthenes' approach to science can be positively interpreted by stating that he tried to dominate the complexities of reality (in facts his appointment at the Library required this kind of skills!) and, for this reason, he had to prove to have talents in a large variety of disciplines. He was capable to understand many things and wanted to use every kind of information which he could achieve. As a consequence he could not be the best expert in anything, but he could play a role in transmitting knowledge from a discipline to another. In facts, as reported by Strabo: Eratosthenes was regarded to be a mathematician

[2]The Suda is a Byzantine encyclopedia, written during the 10th-century after Christ. It is a Greek lexicon, having 30,000 entries and including many drawings copied from ancient sources, sources which have been, unfortunately, subsequently lost. The name derives probably from the Byzantine Greek word "souda", which means "stronghold [of knowledge]". Eustathius, misunderstanding the etymology of the title, declared that Suda was a deformation of the name Suidas, that was his author's name.

among geographers and a geographer among mathematicians[3].

His skills placed Eratosthenes in a very privileged position: he could decide what had to be published becoming a book stored in the library of Alexandria, and therefore, considering the importance of this library, which book could be transmitted to future generations. Without any doubt, Eratosthenes belonged to the *timocratic scientific élite*, i.e. the dominant group of intellectuals of his epoch. Archimedes, who usually could not hide his great self-esteem (see [30, 31]), was, however, obliged to treat with great reverence such an important person. He, therefore, called him a "diligent", "an excellent teacher of philosophy", and "greatly interested in any mathematical investigations that may come your way". Archimedes, as modern scholars are often doing when submitting a paper, writes clearly to the editor-in-chief about its motivations:

> *I am convinced that this* [heuristic method] *is no less useful for finding the proofs of these same theorems. For some things, which first became clear to me by the mechanical method, were afterwards proved geometrically, because their investigation by the said method does not furnish an actual demonstration.*

The reader must remember here that the expression "proved geometrically" is a precise calque of the Greek original expression. It has to be understood, in modern language, as follows: "proved with mathematical rigor". Archimedes has a great standard of mathematical rigor. He states that something is "proven" only when he finds a logically precise sequence of statements which can be deduced, one after the other, from his axioms. A heuristic reasoning is NOT a theorem, for every mathematician since the Greek invention of rigorous mathematics. The use of the word "geometry" in Archimedes' text is simply related to the fact that, in Hellenistic science, the theory of real numbers was formulated in terms of geometrical entities like segments, areas and volumes (see e.g. [32]). The argument of Archimedes continues as follows:

> *For it is easier to supply the proof when we have previously acquired, by the method, some knowledge of the questions than it is to find it without any previous knowledge. That is the reason why, in the case of the theorems, the proofs of which Eudoxus was the first to discover, viz. on the cone and the pyramid, that the cone is one-third* [of the volume] *of the cylinder and the pyramid one-third of the prism having the same base and equal height, no small share of the credit should be given to Democritus, who was the first to state the fact about the said figure, though without proof.*

Archimedes is aware of the importance of both the heuristic, creative invention act which leads to the conjecture of a mathematical result and the technical rigorous demonstration which is needed to state that such a theorem is true. He

[3] This destiny is bounded to modern mathematical physicists: they are neither mathematicians nor physicists. However they can be useful in allowing for the communication among the two groups.

distinguishes between the inventor of a mathematical proof and the discoverer, who is aware of a well-conceived conjecture, whose result is left to be proven. Then he discusses the specific heuristic procedure, based on his understanding of a problem of mechanics, which lead him to calculate the area of a parabolic section:

> *My own experience is also that I discovered the theorem now published, in the same way as the earlier ones* [the theorems conjectured by Democritus and proven by Eudoxus]. *I now wish to describe the method in writing, partly, because I have already spoken about it before, that I may not impress some people as having uttered idle talk*

Archimedes wants to underline that his creative work has to be split into two parts: i) the conjecture of the statement of the theorem, based on a heuristic argument, and ii) the rigorous proof of the theorem, based on a logical procedure, starting from the axioms he has accepted. It has to be remarked here explicitly that Archimedes calculates the aerea of a parabolic section by what will be called later an integration method. For doing so, he needs the rigorous definition of the set of real numbers, which Archimedes attributes to Eudoxus of Cnidus. On the other hand he conjectures that the area of the parabolic section has a certain value by means of an experimental measure. Archimedes, following a habit that is unfortunately too often spread among pure mathematicians, communicated his rigorous proof without any reference to his heuristic mental process. However he had spoken about it while discussing with his colleagues: he feels the need to describe it in a written form. He is doing this in order to keep his reputation of serious scientist, who is not talking in vain. However to keep his own high reputation is not the only reason for which he discloses his way of reasoning:

> *partly because I am convinced that it will prove very useful for mathematics; in fact, I presume there will be some among the present as well as future generations who by means of the method here explained will be enabled to find other theorems which have not yet fallen to our share.*

Archimedes wants to show to future generations how a theorem is conjectured: he is not happy to give the rigorous proof of it, only. As he has not a technique of discovery which can be formally presented to the reader, he explains his own mental process, based on a clear understanding of mechanical phenomena. Finally he gives us the specific technical details concerning his theorem

> *We will now first write down what first became clear to us by the mechanical method, viz. that any segment of an orthotome*[4] *is larger by one-third than the triangle which has the same base and equal height, and thereafter all the things that have become clear in this*

[4] An old name first used by Menaechmus to designate the particular conic section resulting from cutting a right-angled cone by a plane which is perpendicular to its surface, thus producing a parabola

way. At the end of the book we will give the geometrical proofs of the theorems whose propositions we sent you on an earlier occasion.

The few sentences cited above were considered by Heiberg, their modern discoverer, as possibly the most important ones uttered by Archimedes. Archimedes transmits to us the mental process which occurred in his mind during his mathematical creation. Rather seldom such a clear perspective is given in a mathematical text. Hellenistic Mathematics, and also all subsequent mathematical tradition, is characterized and founded on the logical rigor of the presentation. The economy of thought and its precise formulation are considered the prevalent criterion when presenting mathematical results. A mathematical text, since Hellenistic mathematicians, is a sequence of logical conclusions, obtained with correct deduction rules, starting from the accepted hypotheses, conceived in such a way that the theses are related to the hypotheses by a irrefutable reasoning. While this demand of rigor is essential for the development of hard sciences, it is also undoubtedly true that this style of presentation, giving the synthetic final result of the process of demonstration, is ignoring the equally important demand of understanding the reasons which led the mathematician to the presented demonstration and the heuristic method using which this demonstration was found for the first time. Risking to spoil the myth of his own genius, Archimedes reveals spontaneously how himself, before even starting to try to prove his theorems, conjectured their theses and managed to be persuaded that they were true.

3 An epistemological intermezzo: inductivism versus falsificationism

Without any hope to succeed in presenting an exhaustive report of the epistemological knowledge that led us to understand how scientific theories are built, for seek of self-consistency we sketch here those most fundamental ideas that should guide a mathematical physicist in his scientific practice.

We have been sometimes very surprised in discovering that otherwise very gifted scholars may have a too naive vision about the epistemological concepts which are needed for correctly guiding their scientific research. In general, for what concerns the postulation scheme used in Continuum Mechanics we have seen too many presentations in which a series of ad hoc postulates are accepted based on *experimental evidence* or even claiming that they are *induced by experience*. These approaches led to an occlusion of Continuum Mechanics in a stage that was already recognized to be too particular in the works of Gabrio Piola [2–6, 13, 33].

In order to get rid of the limiting scheme of Continuum Mechanics as elaborated by Cauchy and imposed in Engineering Sciences by its undoubted successes in predicting deformative behavior of bodies, it is necessary to resort to a truly falsificationist approach in the comparison of different mathematical models used for *describing* reality.

3.1 Relation between Science and Technology: a view back through History of Science

For the kind of analysis we want to conduct, it is of primary importance to ask what is the effective relationship between Science and Technology. Is there a theory that describes the birth, growth and decay of Scientific Theories and Scientific Technology? To get answers in this direction it is necessary to refer to concepts that are the specific object of History and Philosophy of Science. If thinking about History of Science does not confuse us, because we can easily recognize in it the ordered set of observed facts, discussing Philosophy of Science may induce misunderstandings. We refer to Philosophy of Science as that meta-theory which, by organizing the set of available information about the way in which well-established theories were constructed, tries to supply efficient methodologies apt to formulate new theories. In the perspective of a mathematical physicist, therefore, a Philosophy of Science is indispensable.

But let's go back to the original question that we believe has a basic importance: what is the relationship between the development of an organized Science and the technological progress of a society? The answer to this question is extremely complex, but we can already get a clear idea by considering on an imaginary time line the focal points of human technological development and then, on the same time line, place the cornerstones of scientific development. What would immediately appear is that for about two million years man has used chops and more or less polished stones for hunting, working skins, cutting wood and other subsistence activities. A few thousand years before Christ, man began to build the first instruments. Gradually technological advances have increased, but there has been an incredible acceleration in correspondence with the birth of Hellenistic Science: the ballista, the Syracusia ship, the astronomical calculator of Antikythera, just to name a few. It has to be remarked that the existence of disk of Nebra (we will give details about it in the following) seems to indicate that, albeit we do not have any written evidence about it, the great development of the technology related to the Bronze Age may be related to a first elaboration of a proto-Science.

One can follow this imaginary timeline up to the present day by observing how the relationship between scientific development and technological development is inextricably linked. A society that abandons Science, after a suitable time-delay, goes through three successive phases:

i. it no longer produces any kind of new technological development,

ii. it loses the knowledge related to the use of technological tools developed in a previous era of scientific flowering and

iii. transforms (in the best case) such tools into religious objects.

We will see how this decline of Science, and consequently of Technology, is inexorable when certain conditions are created in a given society. One can possibly explain the fall of Western Roman Empire relating it to the loss of awareness about the importance of Hellenistic Science and the related slower, but equally inexorable, loss of technological capacity. We believe that there is an exemplary case that deserves to be shortly discussed here: we mean the use of gravity aqueduct. Hellenistic hydraulics did know a form of the law that has been named after Bernoulli. This theoretical knowledge allows for the conception and construction of the cheaper pressurized aqueduct. In fact, in Pompei we can see a network of pipes distributing the water in the city with a small local pressurized aqueduct. However, building a large aqueduct is not a very frequent need. In the Pergamon Museum in Berlin important parts of a large pressurized aqueduct serving the Pergamon Acropolis are shown. We do not know when the needed theoretical knowledges of hydraulics were lost: for serving Rome, unfortunately, engineers who ignored hydraulics built gravity aqueducts, causing a large economical loss. A sum of such losses most likely made the difference of the destiny between Western and Eastern Roman Empires. One may consider that for some unknown reason the advanced topographic knowledge needed for building a gravity aqueduct were not lost in the passage between Hellenistic and Roman cultures: the reasons for which Romans did manage to preserve a part of Engineering Sciences (Topography) while loosing another part (Hydraulics) maybe be related to arbitrary choice of a librarian who could not understand the mathematically difficult arguments in Hydraulics while could catch the simpler reasonings used in Topography, probably because this last can be synthesized using drawings and simple Euclidean Geometry.

An interesting philosophical question that arises spontaneously when we try to organize the phenomenology of scientific progress of human societies is to wonder if the path of human history is a progression of stages that has been repeated many times, independently by different groups, in the same order or if each progress has occurred only once and then it has consequently widespread. This distinction between social determinism and diffusionism finds its basis in the thought of Giambattista Vico, who wrote

> *Similar ideas that originate from entire peoples unknown to each other must have a common basis of truth.*

We tend, differently by what appears in Vico's thought, towards a diffusionist approach. This approach explains better the phenomena related to the scientific flowering which occurred in the Renaissance. Is there really anyone who can believe that the Renaissance evolved in a completely autonomous way? Can anyone really continue to deny the very strong influences that Hellenistic thought had on Renaissance thought? And if there are still few who deny such influences, why, instead, are there still so many who deny the importance of Hellenistic Science and even deny it a classification as a truly "modern" Science?

The library of Cardinal Bessarion is the first fundamental part of Marcian Library in Venice and was constituted mainly by Greek codices. Based on the *transport* of Hellenistic Science via Greek manuscripts arriving in Europe, the main characters of Italian Renaissance started the re-discovery of ancient Science not always recognizing their debt towards their sources.

3.2 Approaches to Science: Falsificationism or Inductivism?

In the formulation of a scientific theory at least two alternative approaches can be used. Following the standard nomenclature in the literature, they are called inductivism and falsificationism. We are aware of the fact that more sophisticated conceptual frames have been adopted in Philosophy of Science. However, discussing only these two approaches is enough for our aims. Both of these visions can be traced back throughout the History of Science. As far as we will discuss in the following of this chapter, we are interested in how they were declined in Hellenistic thought, as we will analyze the development and decline of the models introduced for the description of the motion of the planets, and how they were used within the group of scientists who in the XIX century and later developed modern Continuum Mechanics.

As for Hellenistic Science, as we shall see, unfortunately surviving sources are so rare that it is difficult to tell in which form the debate on inductivism and falsificationism took place among Hellenistic scientists. The echoes of this debate, however, are resonating in a significantly later period: Proclus (412-485) discusses the nature of epicycles (we will see below the details of the deferent-epicle model) and asks himself whether they exist or are pure mathematical hypotheses in his treatise Hypotyposis (i.e. Exposition of Astronomical Hypotheses). As we will see, for scientists of the Hellenistic age, as Apollonius of Perga who first introduced planet models using deferents and epicycles, it was obvious that these were simple mathematical objects and that they are not objects in the physical world. They loose every meaning if not contextualized in the model where they were introduced. As Proclus is a post-scientific philosopher, he seems to report about an ancient debate and, being completely unable to fully understand its content, he manages to deny the validity of both positions. However, Proclus claims to be a follower of the philosophical thought of the Platonic school: therefore, he should be able to see a difference between mathematical and physical objects, albeit believing that one can experience, in the world of mathematical ideas, some experiences leading mathematicians to mathematical theorems.

When Platonism is adopted in the development of mathematical thought, then extreme positions are generated. In fact, according to Hardy [34], mathematical platonism is based on the statement

> *Mathematical reality lies outside of us and our function is to discover and observe it and the theorems we prove* [...] *are simply the accounts of our observations.*

According to mathematical Platonism, then, physicists discover physical reality while mathematicians deal with mathematical reality. As we will see with examples taken from both the development of models for the motion of the planets and the development of modern Continuum Mechanics, it is very dangerous to confuse, or even identify, mathematical entities with the physical entities of which they are assumed to be models. Moreover, there are some mathematical entities for which one cannot find any physical correspondence: these mathematical entities are useful only in the logical development of the formulated mathematical model. When one confuses the mathematical model with the physical reality, it may happen that, instead of concluding that the specific model is not suitable to describe physical evidence, one could believe that reality is not self-consistent and may arrive at the conclusion that nature is intrinsically paradoxical. This ontological point of view should be avoided if one wants to have any hope to describe and predict physical phenomenology.

The confusion between models and physical reality is carefully avoided by Platonic mathematicians: therefore, such a philosophical position is not impeaching the needed distinction between mathematical objects and the physical objects they are modeling. Once one has distinguished between mathematical models and real objects, it is easy to confute the so-called inductivist vision of Philosophy of Science.

Inductivism has been considered for too long time as the true scientific method that has to be practiced by diligent scientists. Unfortunately, it is still a commonplace view in many scientific milieux to believe that one can induce from many observations some physical laws, that belong to physical reality and can be established once forever. Such a vision of the scientific method is not efficient and effective to develop scientific theories, as an efficient process like *induction of a physical law* cannot be established. In fact, inductivism is based on the belief that a systematic research approach exists, that involves an inductive reasoning (whatever it may mean) enabling scientists, when applied with due diligence, to *objectively discover* the unique true theory describing every phenomenon. The prescription of inductivism, when examined attentively, presents a very ambiguous clause: the scientist must apply *due diligence*. Therefore, when an induced physical law reveals some limits, naive inductivists are simply stating that the scientist formulating it was not diligent enough. Such a point of view is not at all scientific: how can a scientist know which is the *due diligence* necessary for being sure that his law is "true"? The position of naive inductivists has been ridiculed by Bertrand Russel with his famous anecdote about the inductivist chicken [35, Ch. 6, p. 47]:

> *Domestic animals expect food when they see the person who usually feeds them. We know that all these rather crude expectations of uniformity are liable to be misleading. The man who has fed the chicken every day throughout its life at last wrings its neck instead, showing that more refined views as to the uniformity of nature would have been useful to the chicken.*

In a more picturesque way, Chalmers in [36] reformulates it as follows:

> [We present] *a gruesome example attributed to Bertrand Russell. It concerns a turkey who noted on his first morning at the turkey farm that he was fed at 9 am. After this experience had been repeated daily for several weeks the turkey felt safe, in drawing the conclusion "I am always fed at 9 am". Alas, this conclusion was shown to be false in no uncertain manner when, on Christmas eve, instead of being fed, the turkey's throat was cut. The turkey's argument led it from a*

number of true observations to a false conclusion, clearly indicating the invalidity of the argument from a logical point of view.

More seriously and shortly, but maybe in a more effective way, Einstein also criticizes inductivism:

> *Any amount of experiments may prove that I am right; a single experiment can prove that I am wrong* [A. Einstein, letter to Max Born on the December 4th of 1926].

In conclusion, *the idea that theories can be derived from, or established on the basis of, facts* is a statement with an empty meaning, and we believe that the same use of the world "theory" is not appropriate. In fact, a theory is, etymologically a sequence of statements deduced logically from a conjectured set of postulates. The commonplace statement which we have quoted before should be rephrased by introducing instead the world "physical laws" if one could give a meaning to such an expression.

Inductivism was formulated, in our opinion, while misunderstanding Hellenistic sources that stressed the importance of the experimental verification of formulated mathematical theories. Inductivism was developed during four centuries and Francis Bacon was one of its champions. Western Europe's prevailing epistemological approach, in the époque of Bacon, was the so-called scholasticism. Also scholasticism was based presumably on a misunderstanding of Hellenistic sources: the philosophers of this school believed that, based on preconceived beliefs, one could, without any interrogation of experimental evidence, forecast the behavior of physical phenomena. Clearly, scholasticism was accepting only partially what we presume was the true formulation of ancient falsificationism. The falsificationist approach, which consists in conjecturing a model having the aim of describing a set of observed facts, verifies only a posteriori how much can be predicted on the basis of the assumed conjecture.

In fact, falsificationism bases its analysis of natural phenomena, and the corresponding formulation of theories, on the conjecture of some basic postulates from which the scientist must deduce consequences, to be used, when possible, to predict physical phenomena. Therefore, while the stress of scholasticism was presumably focused on the first part of the process of scientific invention as described by ancient falsificationism and neglected the important required check obtained by experiments, inductivism stressed only on experimental evidence, by loosing the deductive part so highly considered in ancient falsificationism. It is clear that the scholars of Middle Ages, having a partial understanding of their sources, could catch only a part of the original complex epistemological vision. This vision has been completely reconstructed only at the beginning of 20th century, when it was necessary in order to formulate really novel physical theories like Quantum Mechanics or General Relativity.

A falsificationist does not try to induce his postulates, he only checks that all the logical consequences of his postulates, for which this is possible, are verified experimentally. Falsificationism has shown to be extremely advantageous in the advancement of scientific progress, compared to a naive inductivism. We claim that one of the first implicit expositions of falsificationism can be found in Archimedes' *Treatise on Method*, in which the Syracusan scientist provides guidance on how to proceed in conjecturing new theories correctly. If it were not for the fact that modern Science is Archimedes' progeny, we could say that Archimedes has all the characteristics of a modern scientist!

Contrary to what History of Science has shown so far, i.e. that only a scientific knowledge produces advances in Technology (therefore, we claim that the only possible way to produce new technological advances is to develop new theories that allow us to observe phenomena never observed before), unfortunately today scientific progress appears to be stuck in the pointless debate on a *data driven* or *theory driven* Science. This debate represents the modern rephrasing of the debate between inductivism and falsificationism, that seems to have been evoked by Proclus.

Proponents of the *data driven* strategy, strengthened by the fact that today there is a relative overabundance of data available and computing capacity, argue that the description of reality can be simply induced by means of the manipulation of experimentally collected data. We will see, in the following, a fundamental example of how even the modern critical interpretation of Hellenistic Science is sometimes given in a *data driven* key. In fact, while Hipparchus of Nicaea conjectured a priori the precession motion of the rotation axis of the Earth, today's modern inductivists, who are data driven, let us believe that Hipparchus *induced* the precession law from a comparison of the positions of certain stars as measured by him and those reported in a star catalog compiled 150 years before him. We believe, and we will describe extensively the reasoning that leads us to this belief, that, instead, Hipparchus first conjectured Earth's axis precession and only after then, based on his conjecture, explained the discrepancies between the two catalogs. Albeit we do not have the relevant sources available (imagine if we could find Hipparchus counterpart of Archimedes' *On the Method*!), we can suppose that, after having seen the motion of a spinning top (see below for more details), Hipparchus, knowing what he was looking for, checked the star catalog for obtaining a confirmation of his conjecture.

The debate between inductivists and falsificationists is being repeated nowadays, for instance, also in the research field devoted to the invention of new materials with properties which are not observed *spontaneously* (i.e. not too frequently) in nature. In this area, which is also discussed extensively in other chapters of this work, a "data driven" strategy is not only impractical, but also conceptually wrong and economically disadvantageous. Therefore, we claim that an awareness of epistemological basic concepts is needed also in nowadays researchers studying basic problems in Engineering Sciences.

3.3 Underdetermination of Scientific Theories: a problem for Inductivism?

In the conceptual framework we have discussed up to now, when formulating a new theory a fundamental role is played by the basic hypotheses, or physical postulates. In the falsificationist approach, starting from the basic hypotheses (postulates), and using rigorous logical procedures, one can deduce consequences that can be confronted with experimental data. It has no sense wondering a priori whether hypotheses are true or false: hypotheses can be only judged on the basis of the comparison between the whole set of their consequences and available experience. Moreover, hypotheses have to be contextualized in the model for which they are formulated. It is a very common misunderstanding the confusion between the hypotheses of a specific model and the hypotheses of another model treating a different aspect of the same physical system. Also if two models are describing the same physical entity, this does not imply that one has to assume the same hypotheses in both of them, if the phenomena to be described are sufficiently different. We present here some paradigmatic examples of this underdetermination of scientific theories.

We do not believe into the inductivist approach, because, obviously, a collection of phenomena concerning a physical system does not uniquely determine *the true and only scientific theory* to be used for describing it. In fact, and as we have stressed before, the used hypotheses may change when choosing a model or another model for the same physical object. A very famous example of the underdetermination of scientific theories is given by Archimedean study of the mechanical behavior of Oceans.

Let us start from a strong ontological statement, clearly accepted by Archimedes: oceans exists and are always the same physical object where tides occur and on which vessels float! Now Archimedes knows that the phenomena involving the floating of vessels can be described by the model of planar surface of oceans. In facts, Archimedes uses the hypothesis that the surface of seawater is a horizontal plane (in the treatise *On floating bodies*) as a basic one when he wants to establish the stability conditions for ships hull in the vertical configuration. Archimedes had to develop his famous buoyancy law to found this specific theory. However, somewhere else (we conjecture this happened when he was preparing the model for describing tides, that we know has been developed by Seleucus) Archimedes also proves, starting from other postulates, that the surface of the Oceans has to be spherical!

He knew how to use different hypotheses, depending on the different type of phenomena he wanted to describe. Can we find a contradiction between the two models for the surface of Oceans? Is Archimedes, as it is claimed by some modernistic historians of Science, a primitive and confused scholar? In fact, the two visions of Nature, as described by the two Archimedean models, can be reconciled. The floating phenomena of vessels, actually, can be described either by assuming a planar ocean surface or by a spherical surface with a much bigger Earth radius than the dimensions of the vessel. We can, then, easily agree with the fact that a collection of phenomena does not uniquely determine a scientific theory and that the basic hypotheses may change when considering different models formulated for describing different phenomena involving the same physical object.

How can we decide if the Earth surface is not more complicated than a sphere? The ancient Greek observation that one sees at a distance the sails of a ship before seeing its hull can be explained in different ways, attributing to the surface of the Earth different shapes. An application of Occam razor suggests that it is wise to start with the simplest conjecture: that it is a sphere. However, every surface locally similar to a sphere can, in principle, be adopted. Once more naive inductivism seems to find an insurmountable obstacle.

Another useful example is given by the many different models introduced for describing the physical objects planets (and specifically the Earth). The possibility that one can model Earth as a moving material point (as it is done in Celestial Mechanics), or as a rigid sphere (in elementary Astronomical Geography), or as a rigid geoid (in advanced Astronomical Geography), or as a deformable geoid (in Seismology), or as a multi-phase deformable solid (in Geochronology), implies that there are not preferential *true* hypotheses to adopt, but that for a given set of phenomenological evidences a most *suitable* mathematical model is conceivable and that the discussed underdetermination can be solved with a kind of *minimization principle*, that is Occam razor.

In conclusion, we share the belief that (i) the basic postulates of a theory are statements whose truth value can be uniquely posed a priori and (ii) only their being false can be determined once for all. The previous statement is the essence of falsificationist approach, while naive inductivism believes that the basic postulates of a theory can be proven to be true by means of a series of experiments. To believe into inductivism is a (negative) change of perspective dating back to more recent times (i.e. Newton) with respect to the Hellenistic view. This perspective change, we believe, corresponds to a diminution of epistemological awareness.

In fact, Archimedes accepts that a certain theory is valid to describe the phenomena of buoyancy and understands that for this theory to be predictive it is necessary that a certain theorem be true, starting from some basic postulates. So he commits himself to prove this theorem with mathematical rigor.

The example, to which we refer, requires the application of the law of buoyancy and the demonstration of a theorem, which is given by Archimedes by an argument of exhaustion. Archimedes understands that formulating postulates is an important step in the procedure of developing any scientific theory and that experimental evidence cannot be used to prove theorems, that is the consequences of the accepted postulates.

The clarification of the role of mathematical deduction from postulates and of their comparison with experiment represents the main ideas contained in his treatise *On the Method*. The epistemological ideas at the basis of that treatise are manifestly more modern than many contained in works that claim to be milestones in modern Science. Paraphrasing Archimedes, we can say that the fact that the law of buoyancy produces some predictions that can be experimentally verified (using modern language) does not imply that the equality is mathematically true, on the contrary it must be, in fact, proven starting from the mathematical definition of the set of real numbers. Archimedes is confident of the descriptive capacity of his model in explaining buoyancy phenomena. Therefore, he is ready to assume that the entire mathematical architecture needed in the deductive part of his theory is correct, and that the predictivity of his model points the way to a demonstration of the mathematical theorem that *must be true*.

In doing so, we believe that Archimedes proves to be a falsificationist. Moreover, he is so aware of the importance of his Method that, as we have already previously remarked, he claims:

> *I am persuaded that it* [the Method of Mechanical Theorems] *will be of no little service to mathematics; for I apprehend that some, either of my contemporaries or of my successors, will, by means of the method when once established, be able to discover other theorems in addition, which have not yet occurred to me.*

History of Science teaches: developing mathematical models for describing new phenomena can lead to unexpectedly useful results not only in inventing new technological artifact, and predicting the existence of new phenomena, but also in conjecturing new mathematical theorems. This point will be made clearer in the next sections.

4 From the world reality to its mathematical model and from the model to the replacement of the world reality

In this section we present two paradigmatic cases of how several times human society has seen the birth and subsequent decline of Science. A very interesting aspect lies in the fact that the state of decline is generally not universally recognized except by a few voices that are however isolated and, if possible, silenced. The picture that comes out from the analysis of many cases of decline that have affected human society is disconcerting: it could seem that this decline is the result of an extremely organized operation rather than the result of a series of unhappy choices, of either political or social nature. The question arises spontaneously: who would benefit most from the decline of a scientific society? Who would have the courage to condemn the human society to a sort of Dark Ages in order to favor their own interest?

The answer is not uniquely determined. Certainly when in human society a few groups of unscrupulous individuals assume the leadership and replace in power people who are prepared and work for the common good, the decline is already at a very advanced stage. One aspect which is common to all moments of decline is the relative importance that bureaucrats acquire. Bureaucrats who should limit themselves to facilitating the choices of politicians replace them and ensure that society remains entangled in useless discussions. When, in the late Byzantine era, the highest scientific-philosophical discussion of the intelligentsia of the time concerned the sex of angels, society had already been in decline since long time and the conquest of Constantinople with the consequent collapse of the Eastern Roman Empire in 1453 represented only the formal end of an era that had already ended long before.

The process that determines the decline of a society does not merely ensure its end at a given historical moment, but often also ensures that not enough traces of its civilization survive to determine a new cultural and scientific flowering at a later time. This results in a veritable erasure of certain theories or of the name of their founders.

A notable example that reaches us from Greek antiquity is given by the case of Archytas of Tarentum, who was several times *strategos* (i.e. general) of Tarentum (and therefore his name could not be erased completely from history). He was the first to introduce the Principle of Virtual Work in the study of mechanical systems, but knowledge of this was lost until a few years ago when the treatise *Mechanica Problemata* historically attributed to Aristotle was recognized to be likely authored by Archytas, according to T.N. Winter (2007). On the other hand, it was not lost the information that he had invented a mechanical bird and a toy called ratchet. It is interesting to see that not only his name was erased from the *Mechanica Problemata* (which, by the way, could be an exercise book associated with a much deeper theoretical text), but it was transmitted to us only that his main contribution in Mechanics was the invention of toys. Instead, we believe that he was considering these toys as a way for explain the basic mechanical principles exactly as Heron of Alexandria did later in his *Mechanica* and *Automata*. The process of cancellation is systematic: not only it does eliminate all original sources that it can, but when it cannot manage to eliminate them altogether it makes them sound less authoritative. It is not easy to establish if the erasure process which cancels the name of great scientists and deforms or removes completely their theories is conscious or a consequence of the lack of intelligence and capacity of understanding. This dilemma appears also when discussing the motivations of those politicians mentioned before, whose choices produce the cultural and scientific collapse of the societies that they lead. Most likely the behavior of both scholars and politicians whose disastrous choices were mentioned before can be described by a famous Friedrich Schiller's quote[5]:

> *Against stupidity the very gods themselves contend in vain.*

One of the most frequent phenomena occurring in the phase of degeneration of the scientific culture in a social group consists in the systematic confusion of a mathematical model with the physical object that this mathematical model is aimed to describe. Of course, this confusion is deadly because it poses a series of apparent paradoxes which may lead to believe that the predictive limits of the model represent instead an intrinsic self-contradicting nature of reality. The destructive ontological consequences of these phenomena may lead to a violent reaction against the process of mathematical modeling, that can be exemplified by the skeptic philosophy that led Sextus Empiricus to abjure Hellenistic Science.

We can recognize a repetitive pattern of growth, decline and collapse of scientific theories and scientific cultures, so that we tend to generalize the anal-

[5] *Die Jungfrau von Orleans* (The Maid of Orleans) (1801), Act III, sc. vi (as translated by Anna Swanwick)

ysis by Giambattista Vico, originally limited to the cyclic repetition of social structures, also to history of Science.

In order to show how the decline of a scientific society generally occurs, we propose in the following two examples of models that, in the progress of time, have been confused with reality itself and have, therefore, generated atrocious misunderstandings. The first case consists in the observation of how the mathematical model for the motion of the planets, formulated in an extremely accurate way and also by means of advanced mathematics in the Hellenistic age, has been abused and completely misunderstood until producing in the Middle Ages the idea that the planets actually moved on metallic guides placed in the heavens. The second case that we will deal with is that of Continuum Mechanics, where starting from a certain époque, the concept of force, which was introduced only to simplify the mathematical formulation of the Principle of Virtual Work, has assumed a completely unjustified fundamental role in the postulation of basics Mechanics principles. We observe here that it is possible to recognize a process of *materialization* or *transformation into a real object* for the completely abstract concept of force. In a kind of Platonistic delirium many scholars managed to persuade themselves, and to persuade their pupils, that forces are real objects that one can meet in everyday life: the resulting confusion between physical objects and mathematical objects used in a model for describing real world phenomena is extremely misleading. Those who believe in the *reality* of forces want to give at any cost to this object a wrong ontological essence.

5 Reconstruction, partly conjectural, of the birth and decline of the mathematical models for planetary motion

We now want to mention a reconstruction, clearly partly conjectural, of the evolution of the mathematical models for the motion of the planets. It is necessary to make two premises: (1) the purpose of what we will describe is not the in-depth historical study of given scientific theories (for this there are several texts available in the literature [32,37]), but rather to show a sociological aspect of the transmission of scientific culture, which, of course, can be studied only by resorting to the development of the models in non-negligible periods of time; (2) the reconstruction that we will present of the evolution of the motion of the planets is obviously conjectural, in the sense that not all sources are available, but, from the few sources that have come down to us and from secondary sources, it is possible to conjecture the scientific panorama of the Hellenistic age to obtain a vision about Hellenistic Science that is, in many aspects, really surprising.

5.1 An example of precursory proto-model before Hellenistic Astronomy

Before properly analyzing Greek astronomy, it is appropriate that we mention an object dating back to the Bronze Age recently found near the German town of Nebra. It is a bronze disc with gold applications representing the sky, which was most likely used in the period 2100 BC - 1700 BC. Specifically one can recognize the crescent Moon, the Pleiades and a disk that could be the Full Moon or the Sun. Two arches are affixed to the edges of the disk. In more recent times it has been added a small arch that could represent a solar boat, typical of a religious representation and also found in other cultures such as the Egyptian one.

It has been conjectured that the disk could be used to precisely determine the equinoxes and solstices, aligning it with the stars at certain times of the year and taking into account the orography of the place where it was found. So it would be a rudimentary scientific instrument used for determining the calendar of agricultural activities: it is therefore one of the first available examples of a technological tool developed on the basis of a predictive model about the universe but used for practical applications. The subsequent affixing of the solar boat suggests that the scientific instrument has been transformed into an object of cult and then was finally buried in a tomb. The story of the disk of Nebra is the story of a scientific society, obviously in its embryonic state, that arises and produces useful instruments and then declines making what is no longer understood to become a religious cult. The fact that the fate of a scientific instrument, which is no longer useful because it has been transformed into a cultic object, is the tomb is very explanatory.

The disk of Nebra gives a strong support to the vision of history of Science that considers cyclic cultural declines as frequent social phenomena. It supports Giambattista Vico's vision of cyclicality of social phenomena and completely falsifies the belief that human progress is only proceeding towards higher cultural consciousness.

5.2 Eudoxus and the model of homocentric spheres

The first known scientific model describing the motion of the planets is due to Eudoxus of Knidos (408 BC - 355 BC). Eudoxus was a mathematician and astronomer. It is one of the fathers of mathematics. Pupil of Archytas of Tarentum, among other things he studied the problem of finding the algorithm (with ruler and compass) for the duplication of the cube. The problem of the duplication of the cube is an absolutely non-trivial problem, since to be treated properly one must master the concept of irrational numbers, that before Eudoxus most likely had not been developed. It has to be remarked that, when Pythagoreans discovered that the hypothenuse of an isosceles right triangle is incommensurable with the catheti, the first reaction was to believe that nature was paradoxical. In fact, Pythagoreans did confuse the mathematical model *rational numbers* with the concept of *length of a segment*: when it was proven that the above-mentioned hypothenuse could not be represented by a fraction they were led to believe that such hypothenuse did not exist. The reader is invited to consider this as a paradigmatic example of the disastrous potential consequences of the epistemological and ontological mistake which is intrinsic in the confusion of a mathematical model with the physical object that one intends to describe.

It is not a coincidence, then, that Archimedes attributes to Eudoxus the invention of the concept of real number in its geometric definition. Unfortunately, all Eudoxus' works are lost: we have only secondary sources. But these are enough to give us an idea of the level of depth of Eudoxus' discoveries. Among the secondary sources, we recall the treatise of Theodosius of Bithynia *Sphaericae*, which is probably based on his work. Of his other works we have received only the titles: *Eclipses of the Sun*, *Octaeterides* (solar lunar cycle of eight years), *Phenomena* and *Entropon* (spherical astronomy based on observations made in Egypt and Cnidus), *In motion.* As mechanicians we were ready to pay a very high price for having a copy of this last text, as it could give us a clear vision of the first true scientific stage of our discipline and could guide us in the development of novel models. Eudoxus' passion for astronomy was not, of course, only theoretical, but had significant practical implications and, in fact, he built an astronomical observatory.

Eudoxus' fame is related to the model of homocentric spheres. This model describes a universe divided into spheres having a single center of rotation. At the center Eudoxus put the Earth surrounded by spheres in uniform circular motion. The outermost sphere contained the fixed stars. On the other spheres moved the planets.

To better understand the mechanics of the model of Eudoxus, we use the words of G. V. Schiaparelli [38]:

> "*Eudoxus thus imagined, almost as Plato had done before him, that every celestial body was set in motion by a sphere revolving over two poles, and endowed with uniform rotation; he further supposed that the body was attached to a point of the equator of this sphere, so as to describe, during the rotation, a maximum circle, placed in the plane perpendicular to the axis of rotation of the same. To account for the variations in the speed of the planets, their retrograde motion, and their deviation to the right and left in the direction of latitude, this hypothesis was not sufficient, and it was necessary to suppose that the planet was moved by several movements analogous to the first, which overlapped and produced that unique movement, apparently irregular, which is what is observed. Eudoxus therefore established that the poles of the sphere carrying the planet were not immobile, but were carried by a larger sphere, concentric to the first, rotating itself in turn with uniform motion and with its own speed around two poles different from the first ones. And since even with this supposition it was not possible to represent the observations of any of the seven celestial bodies, Eudoxus attached the poles of the second*

sphere inside a third one, concentric to the first two and larger than them, to which he also attributed other poles and another speed of its own. And where three spheres were not enough, he added a fourth sphere, including in itself the first three, carrying in itself the two poles of the third, and also rotating with its own speed around its own poles. And examining the effects of these movements combined, Eudoxus found that, choosing conveniently the positions of the poles and the speeds of rotation, the movements of the Sun and Moon could be represented well, assuming each of them carried by three spheres; the more variegated movements of the planets he found required four spheres each. The driving spheres of each celestial body he assumed to be independent of those that served to move the others. [...]

Thus the total number of moving spheres was 26, plus one for the fixed stars. What was the cause of these rotating movements, and how they communicated from one sphere to another, is not found that Eudoxus had looked for; nor what was the material and size of the spheres themselves; nor what were their diameters and their intervals. [...] *Eudoxus therefore totally omitted to research what did not matter to his main problem, the geometrical representation of phenomena; and in this we see another proof of his sober and rigorous genius. He did not care at all to connect the driving spheres with those of the planet immediately above and the planet immediately below, and assumed that the spheres involved in the movement of each planet formed an isolated system independent of the rest. In short, everything leads to believe that the spheres were for him the elements of a mathematical hypothesis, not physical entities; from which he was wrongly reproached for having closed the universe in crystal vaults, and for having multiplied them without necessity.*"

Eudoxus was not a mere observer of the sky: certainly it was by observing the sky that he formulated his conjecture at the basis of the model of the homocentric spheres. In fact, he was a great mathematician: this last characterization leads us to conjecture that he was probably aware that his model was not reality, but only an attempt to describe it. This conjecture is strongly supported by the recognition found in Archimedes sources about Eudoxus invention of irrational numbers: only a sophisticated epistemological understanding could have led Eudoxus to his solution of Pythagoreans apparent paradox. It is significant that many scientists even today are unable to distinguish their model from the reality they claim to describe it.

The geocentric model of Eudoxus did not succeed in any case to explain completely the planets retrograde motions and also failed to give an explanation of the variation of brightness of the planets during their motion (which instead is obvious if we consider that the distance of the given planet from the Earth is variable in time). Remaining within a geocentric model, the system was refined by Apollonius of Perga (262 BC - 190 BC) who first introduced the concept of deferents and epicycles (which we will discuss in more detail when we present the algorithm of Claudius Ptolemy). Apollonius considered the motion of the planets as a composition of several uniform circular motions and in this way he was able to approximate the retrograde motions and to give a convincing explanation of the variation of apparent brightness. Also in this case, as for Eudoxus, we can say with some confidence that the model with deferents and epicycles was perceived by Apollonius as a mere mathematical model and that he was not confusing his model with physical reality. Unfortunately, the same cannot be said about Ptolemy.

5.3 Aristarchus: an ancient Copernicus? Or more likely Copernicus is the modern Aristarchus?

The progress in the development of a model for the description of the motion of the planets obtained by Eudoxus is the basis of the huge advances made by Aristarchus of Samos (310 BC - 230 BC). As we mentioned in the previous section, probably Eudoxus knew that his model did not coincide with reality, but that is was only a description of it, a certainly imperfect and obviously perfectible description: the attempt to describe in some way the retrograde motion of the planets by adding extra spheres represents the most evident proof that Eudoxus had a clear idea of the concept of successive (mathematical) approximations of reality. This idea will be fully developed by the sources of Ptolemy as it is evident by inspecting his computation method based on the introduction of deferents and epicycles. If one wants to build a model to describe the motion of the planets, the first step is to obtain kinematical estimates that are consistent (if not overlapping) with observational data. This is what Eudoxus did. Aristarchus goes a step further and introduces the first heliocentric model. He wonders how well the representation of the cosmos given by Eudoxus closely describes reality. Certainly today everyone should be able to agree with the fact that to pass from the geocentric model to the heliocentric one is a simple change of reference and that, once fixed the correct transformation from a reference to the other, there is absolutely no difference in using one reference or the other one. Actually, another possible, if not preferable, choice would be to place the reference in the center of mass of the solar system and consider the motion of all celestial bodies, including the Sun, around this center of mass. In fact, for the scientist of the third century BC the change of reference is an absolutely not trivial conceptual step. We will see how in the Archimedes' planetary stolen by Marcellus and described by Cicero this change of observer was included in the mechanism.

Let us try to reconstruct the various stages that occur in the research of the Hellenistic scientist to arrive at the heliocentric model. The first observation, the most obvious one, concerns the motion of the Sun and Moon, which describe an arc in the sky during the day and the night. We consider already overcome any kind of religious conception that can come out from such observations and we consider already established the knowledge of the sphericity of the Earth (since Parmenides onwards this was well known to the Greeks!). Based simply on the observations of the positions of the Sun and Moon, it is then licit, for the

Hellenistic scientist, to imagine that these two celestial bodies rotate around the Earth, which instead remains fixed. The big jump in quality of mathematical modeling is made in the attempt to explain the retrograde motions of the planets (to observe and record which quite advanced technologies are already required, because it is impossible to think that with the bare eye one can record with precision the positions of all planets and constellations). A second fundamental observation in the path towards heliocentrism concerns the motion of the fixed stars, which, without apparently changing their inter-distance on the sky, rotate all together. As we will see, the fixed stars constitute a problem for heliocentrism (but Aristarchus responds extremely lucidly to the objection made to him, see below).

So, this is the picture from which the Hellenistic scientist starts:

i. Sun and Moon follow arcs of circumference;

ii. planets show regular and retrograde motions;

iii. fixed stars have an immutable reciprocal inter-distance on the *celestial sphere* that rotates instead on a yearly basis cycle.

The phenomena (i) and (iii) are perfectly described by Eudoxus' model of homocentric spheres. Retrograde motions require a complexified explanation by means of various spheres, with contained relative motions, associated to the same celestial body. To be predictive, Eudoxus' model becomes very cumbersome. In addition, the tendency to the search of the most economical logical reasoning, typical of the Hellenistic scientist, who was trained on Euclid's Geometry and therefore is accustomed to reasoning as simple as possible, cannot explain why a few celestial bodies (planets) behave differently from the other celestial bodies and go back and forth in the sky. We can imagine Aristarchus' astonishment when he realizes that fixing the reference system on the Sun and not on the Earth, the motions of the planets become all nearly-circular (or possibly elliptical): from a complex and cumbersome description, modified *ad hoc* for each celestial body, this Hellenistic scientist is passing to a unified description that treats all the motions of the celestial bodies in the same way. Once heliocentrism is introduced, it will not be possible anymore to come back to other models!

It remains to be settled, in the proposed model, the question of the fixed stars: if the observation is made from the Earth, which according to the heliocentric model is itself in rotation around the Sun, why should the fixed stars appear to have a fixed relative distance? Aristarchus, who, like all his contemporary scientists, knew deeply Geometry, answered in an ingenious and at the same time obvious way: the distance between the Earth and the fixed stars is enormously greater than the diameter of the Earth's orbit, so that, for what concerns our measurements of relative distances of very distant stars, it makes absolutely no difference to fix the observer reference on the Earth or on the Sun. This will be understood again in modern age with Giordano Bruno and Galileo Galilei only.

Obviously, as we will underline in the following discussion about Hipparchus of Nicea, the relative positions between the so-called fixed stars are not at all immutable, but vary on a time scale much longer than the life of a man because of their natural motion. The reason for the very slow variation of the apparent-from-Earth relative distances is related to the enormous distance between the Solar System and these stars when compared with the diameter of the Solar System: exactly the same explanation given by Aristarchus to establish that heliocentrism and geocentrism are equivalent models for what concerns the description of the phenomenology concerning the motion of fixed stars. We believe that this explanation must have been obviously understandable for those who first came to formulate the heliocentric theory, but we have no sources available to know Aristarchus's thought regarding this issue.

Unfortunately the work of Aristarchus on the heliocentric theory has been lost and we have available only some fragments reported by secondary sources. The only work which survived is *On the dimensions and distances of the Sun and Moon*. In this work Aristarchus gives another proof of the high level reached by Hellenistic science. With an extremely simple reasoning he succeeds in deducing dimensions and distances ratios from the Earth to the Moon and to the Sun based on the powerful results of Euclidean Geometry and his own results which are based on trigonometric functions.

The whole reasoning of Aristarchus is based on the fact that, when the Moon is in quadrature, i.e. it is illuminated by half, it forms a right triangle with the Earth and the Sun. By measuring in this condition the angle between the Earth-Sun direction and the Earth-Moon direction it is possible to calculate the ratio between their distances using trigonometric arguments.

To calculate dimensions and distance ratios, Aristarchus is forced to invent a way to approximate the calculation of the tangent of the angle. The tangent of an angle is a function that assumes values throughout the whole set of the real numbers, eventually diverging. In fact, as the angle approaches a right angle, the tangent function tends to diverge, i.e. small changes in the angle correspond to huge changes in its tangent. This implies that if the angle in question is almost a right angle then small errors in the measurement of the angle produce large errors in the calculation of its tangent and therefore in the estimation of distance ratios. The estimate of Aristarchus was, in fact, wrong by several orders of magnitude. However the algorithm invented by Aristarchus for the calculation of the tangent of an angle is correct and it will be very useful for the subsequent development of Hellenistic Science (and of Science *tout court*).

A final note on how many paradoxes may arise while describing the process of transmission of Science is needed. In many modern texts Aristarchus, who first introduced the heliocentric model, is referred to as the ancient Copernicus (who lived almost two thousand years later). As we have repeatedly seen in other chapters of this work, often in the history of Science those who come after claim authorship of an idea, even if this idea was developed by others long before. In the present case, obviously, it was not Copernicus, who probably knew very well the works of Hellenistic Science, to claim the paternity of heliocentrism. In fact, Copernicus clearly attributes to Aristarchus the formulation of such important mathematical hypothesis (see [31]). The causes of this absurd misunderstanding are to be found in the works of modern scholars. Why this reversal of ideas? Why not calling Copernicus *the modern Aristarchus* but, instead, doing the opposite? It may appears as if Aristarchus had in some way wanted to refer to the ideas of Copernicus. The reasons of this aberrant time-reversal are found, in our opinion, in that very modern attitude that sees with extreme disregard the ancient Science (and indeed, many contemporary scholars warn that one should never speak of *science* in antiquity!) and that wants to show how well we can manage ignoring our past. But it should be considered that if removed from the shoulders of giants the dwarfs will fall into the void.

5.4 A mature Science is sometimes too complex to be transmitted to posterity: Hipparchus's explanation of the precession of the equinoxes

Once the heliocentric model has been acquired, Hellenistic Science continued to refine its models by focusing on the study of further available phenomenology. There is, as we have extensively emphasized in the section on Philosophy of Science and Epistemology, an important requirement that a theory must fulfill: not only it must be able to reproduce available phenomenology, but also it has to allow, giving directions to experimental research, for new discoveries by indicating where and how new measurements have to be made. Precisely framed in this panorama, Hipparchus of Nicaea (200 BC - 120 BC) is the first who was able to accurately predict the eclipses of the Sun and the Moon and, demonstrating profound and pronounced skills as a mathematical physicist, as we would say today, he could explain the discrepancies found between the star catalog compiled at the turn of the fourth and third centuries BC by Timocharis (of Alexandria) and Aristyllus (which were based on previous measurements of the Babylonian Chaldeans) and his own star catalog: indeed, between the two catalogs there is a time-lapse of about 150 years and, for this reason, the apparent positions of the stars on the sky show small variations.

Aristarchus, following the hypothesis first suggested by Heraclides Ponticus (c. 390 - c. 310 BC), whose works are however lost, had attributed to the Earth, in addition to the motion of revolution around the Sun, also a motion of rotation around its own axis. He had, moreover, established that, to take into account the alternation of the seasons, it was sufficient that the axis of the Earth's rotation was inclined with respect to the plane of the orbit around the Sun (also known as the *ecliptic* plane). While Aristarchus probably had no idea that the direction of this axis was not constant in time, Hipparchus of Nicaea conjectured the presence of one between the two motions nowadays attributed to the Earth's axis. Hipparchus, in fact, introduced the Earth's axis precession motion, which consists in the rotation of the Earth's axis around the normal to the ecliptic plane: today we also introduce the nutation motion, which consists in a further periodic oscillation of the Earth's axis during the precession motion.

The reasons why Aristarchus assumed the inclination of the Earth's axis should be nowadays part of general culture, albeit they are not at all trivial, as it is needed to explain the alternation of seasons. On the other hand understanding how Hipparchus was able to deduce the motion of precession is an extremely challenging question, which deserves some explanations here.

As we said before, Hipparchus compares two stellar catalogs, that of Timocharis and Aristyllus (based on data already collected by the Babylonian Chaldeans) and his own. The two catalogs have differences in the measured position of some stars (for example Spica). It is possible that Hipparchus formulated his hypothesis of Earth's axis precession from the discrepancies between these measurements. We believe that such a precise hypothesis cannot be the result of this comparison alone, without the aid of a complex modeling procedure and postulation. In fact, once it has been established by Aristarchus that the Earth rotates around its axis and that this axis is inclined with respect to the ecliptic plane, the Hellenistic scientist probably tried to formulate a model of the motion of the planet Earth around its axis, by conceptually separating this motion from that of the rest of the universe. If this simplifying hypothesis is well-grounded, then the model-seeking scientist will as a first step attempt to represent the Earth's axis motion as a superposition of simpler motions. In this aspect, Hipparchus works in continuity with the Hellenistic tradition that represents celestial motions using sub-sequent epicycles. By conceptually isolating the Earth in its motion, it is likely that Hipparchus could have established a parallelism with the motion of a spinning top. It is well attested the use of spinning tops in Hellenistic époque and possibly earlier. Callimachus from Cyrene (about 310/305 - 240 BC) reports the use of spinning tops as toys in his first Epigram [Call. Epigr. 1, 9-10]:

> *Those, some children, played with rapid spinning tops twirling them*
> *in the wide crossroads.*

It is also attested, as proven by the role of Archytas (about 435/410 - 360/350 BC) as inventor of pedagogical toys for children, that Hellenistic scientists aimed to exemplify physical phenomena by means of toys, a tradition which was also continued by Heron of Alexandria (about 10 AD - about 70 AD). It can be assumed, therefore, that Hipparchus, knowing that analogous mathematical descriptions can be used to describe different physical phenomena (today, following Feynman [39], we would say that the same equations may model different phenomena), was bound to attempt the description of the physical system "Earth rotating around an inclined axis" using the knowledge acquired in the already known areas of the Science of his time, i.e. "spinning top rotating around its axis". It is widely known that the Greeks knew the spinning top even before Callimachus: in the VII book of the *Iliad* Homer (late eighth or early seventh century BC) describes the motion of a stone thrown by Ajax Telamonius against Hector as the motion of a spinning top

> *An even bigger stone* [...]
> *Telamonius grasped and his strong*
> *right hand twirled it like a stone thrown from a slingshot.*

If Homer could describe a stone thrown by Ajax as a spinning top, why Hipparchus could not think of modeling the Earth rotating around its axis as a spinning top? Which is exactly what Maxwell will do in his treatise on spinning tops [40]. If one observes a spinning top rotating then he will see: (i) the rotation of the spinning top around its axis, (ii) the direction of the axis changing in time (precession); (iii) the variation of the inclination of the axis due to a certain oscillation (nutation).

The question we have to ask ourselves now is: is it easier to deduce the precession motion by observing the discordance of the measures or to conjecture it by observing the motion of a spinning top thrown by children playing in the street?

It is attested that Hipparchus did conjecture the nutation motion of the Earth's axis. We claim that the genius of Hipparchus consists in imagining the similitude between the spinning top and the Earth and, consequently, in interpreting the discrepancies between the measures reported in the two catalogs not as an indication that the oldest measures could be wrong but as a proof that the Earth could actually be described as a spinning top. It is clear that an essential prerequisite for the advancement of knowledge is that one generation of scientist can rely on the results obtained and transmitted to them by the previous generation. It is therefore to be blamed the modernistic attitude of considering everything coming from the past as unavoidably primitive.

The hypothesis of Hipparchus, contained in his lost work *On the displacement of the solstitial and equinoctial signs*, is applied to the analysis of the longitude of the apparent position of the star Spica during a lunar eclipse. The method adopted by Hipparchus to measure the longitude is known because it was reported by Claudius Ptolemy (c. 100 - c. 170 AD) in his *Almagest.* After the measurement, Hipparchus compared it with the longitude of Spica reported in the catalog of Timocharis and Aristyllus and he noted that this longitude had varied by 2° in about 150 years. From this observation, he made the hypothesis that the fixed stars have shifted with time and estimated a precession of 48" per year. It is remarkable that the precession measured by Hipparchus with the instruments of his epoch is so close to the value measured with today's instruments and expressed as 50.26" per year. It is singular that Hipparchus's estimate is also considerably better than that obtained by Claudius Ptolemy (36" per year) about three centuries later.

The measurements made by Hipparchus to validate his hypothesis of precession of the equinoctial points obviously require the use of instruments, both theoretical and practical, which are extremely accurate. This is how trigonometry and the astrolabe were born. As far as trigonometry is concerned, Aristarchus had already introduced some basic concepts, very much linked to the formulation in terms of Euclidean geometry. In Hipparchus, in fact, we find trigonometry in its modern formulation, except for the use of a different symbology. In fact, the symbology used in modern times, as it is well known, was introduced only by Euler (1707 - 1783 AD).

5.5 Other achievements of Hellenistic Science in the study of the motion of the planets: Seleucus' explanation of ocean tides and the Antikythera calculator

So far we have described the genesis and development of the heliocentric model, but except for the indirect evidence we have mentioned, neither Aristarchus nor Hipparchus had given a *demonstration* of it. According to Plutarch (46 AD - after 119 AD), Seleucus of Seleucia (floruit 150 BC) gave a formal proof of the heliocentric theory. We believe that Plutarch with the word *demonstration* meant the deduction from more fundamental postulates. If our interpretation is correct this could imply that Seleucus had invented a form of dynamics. We will see that Middle Ages echoes of dynamical theories seem to support our conjecture. Another indirect support for it can be found in the explanation, also attributed to Seleucus, of the complex phenomenology involved in ocean tides.

In fact, from a reconstruction based on secondary sources (because even of Seleucus nothing has come down to us) it can be said that the greatest contribution of Seleucus to Hellenistic Science consists in the in-depth study of the tides. Now, while it can be simply understood that the tides are related to the combined interaction of Sun, Moon and Earth, the specific phenomenology, especially in its quantitative aspects, requires a very detailed analysis. Indeed, if one wants to reproduce with some accuracy the experimental observations, the extremely simplified vision where the tidal phenomenon is described by static interactions with celestial bodies is not sufficient. Today we use an extremely complex model based on a dynamic approach, introduced by Laplace (1749-1827), which takes into account also the inertial effect of the ocean motion relative to the Earth. In that formulation, the well-known Coriolis force needs also to be introduced.

An interesting aspect resulting from the few secondary sources of Seleucus' thought is that he related tides not only to the position of the Moon and the Sun, but also to the motions of the Earth. The main sources from which we get information about Seleucus are Strabo (64/63 BC - 24 AD) and Aetius (1st or 2nd century AD), and the latter reports, in an extremely confused way, this idea, which in some ways recalls the dynamic model of Laplace. In this regard, Galileo Galilei (1564-1642) had already tried to give a dynamic interpretation of the phenomenon, but producing not very clear results. We believe that both Galilei and Laplace were at least inspired by the words of Seleucus (probably not by the confusing version of Aetius, but by another clearer source that has not reached us). Some authors, with philological evidence, have tried to interpret the text reported by Aetius and to relate it to the information referred by Plutarch about the presumed demonstration of the heliocentric theory presented by Seleucus, but we will not delve here into this subject.

The scientific progress of the Hellenistic age was not only theoretical but also had strong practical implications. One of the most paradigmatic proofs of the technological development induced by the theoretical advancement of Hellenistic astronomy is represented by the calculator of Antikythera (150-100 BC). This famous astronomical calculator, which even for the complexity of the gears that compose it gives fundamental information on the high level reached by Greek metallurgy, was able to predict an enormous number of celestial phenomena, as well as provide a series of calendars. By turning the crank on the side, it was not only possible to calculate the exact position of each planet and the phases of the moon, but also the eclipses of the Sun and Moon. Since, of course, information about the positions of the planets was given relative to the latitude of a chosen point on the surface of the Earth, some have speculated that, using a sort of inverse method, the calculator could be used on sea voyages to estimate latitude based on a comparison of the positions of the planets in the sky and those determined using the astronomical calculator.

The astronomical calculator of Antikythera shows the position of the planets in a reference centered on the Earth. In this case the choice of the geocentric model is justified by the fact that the scientific instrument has a specific purpose and, if it is true the hypothesis that the calculator provided the latitude by comparison with the sky, then it is logical that the represented system be geocentric. Obviously, in order to describe the complexity of the apparent motions of the celestial bodies in the geocentric system it was necessary to have an extremely precise and reliable computational algorithm. The algorithm used and realized by means of numerous gears was that due to Apollonius of Perga, who decomposed the motion of the planets in circular motions on deferents and epicycles. For each planet the Antikythera calculator has a series of gears for the deferent and for the various epicycles.

The discovery of the astronomical calculator of Antikythera has shown us an aspect of the geocentric system that is rarely emphasized. To the question of why the ancients had begun to study the sky and the motion of celestial bodies, the right answer is not, as it is often trivially suggested, the wonder that the uncultured ancient man felt in observing the starry sky. This is a romantic view that we should learn to circumstantiate. The fundamental reason why it was necessary to study the sky lies in the fact that until before the invention of the compass this was the only reliable way to get orientation. So it is also clear why, although it was already clear with Aristarchus that the heliocentric system was more effective in describing phenomena than the heliocentric one, the geocentric description of planet motions has never been abandoned and, indeed, has been gradually refined: it is absolutely necessary to obtain precise estimates of the positions of the planets in the reference centered on the Earth and thus be able to orient. In short, we could say that the Hellenistic scientists knew very well that at the center of the planetary system there was the Sun, but they needed to put the Earth at the center of the solar system for using the theory to obtain practical results.

Using modern language, the study of heliocentric theory represents pure research, while the study of apparent positions of celestial bodies in a geocentric reference frame represents applied research. Archimedes (287 BC - 212 BC) did succeed in making theory and practice dialogue fruitfully and it is not a coincidence if in the famous planetarium belonging to Archimedes, as Cicero reports, one could, depending on the needs, fix the Sun or the Earth and observe directly the motions of the planets from the heliocentric view-point or from the geocentric one.

5.6 The death of Archimedes as a metaphor of the beginning of the end: the slow decline leading to Dark Ages did begin with the end of Hellenistic Science

Archimedes was one of the greatest scientists and mathematicians of human history. In several parts of this work we have spoken of his outstanding scientific discoveries and especially of his way of approaching the scientific research, which, while remaining strongly connected to physical reality, had the merit of being lucidly formulated in precise and rigorous mathematical terms. As for the topic we discuss in this section, we limit our attention to two aspects, one of technical and the other of historical nature, which are related to the death of the great scientist, after the end of the Roman siege of Syracuse in 212 BC led by consul Marcellus. The technical aspect is reported by Cicero (and we have mentioned it previously): Marcellus brought in the booty of war taken in Syracuse the famous planetarium of Archimedes. Cicero speaks of this planetarium several times, in the *De Re publica* and in the *Tusculanae Disputationes*. In the latter work he reports:

> *"Nam cum Archimedes lunae solis quinque errantium motus in sphaeram inligavit, effecit idem quod ille, qui in Timaeo mundum aedificavit, Platonis deus, ut tarditate et celeritate dissimillimos motus una regeret conversio. Quod si in hoc mundo fieri sine deo non potest, ne in sphaera quidem eosdem motus Archimedes sine divino ingenio potuisset imitari."* [Cicero, Tusculanae Disputationes I, 63]
>
> "In fact, when Archimedes bound in a sphere the motions of the Moon, the Sun, and the five errant planets, he obtained the same result as [the Demiurge] who in Timaeus constructed the universe, i.e. the Plato's god, so that a single revolution governed motions very different from each other in slowness and speed. If it is not possible for this to happen in this world without the intervention of a god, certainly not even in his sphere Archimedes would have been able to imitate the same movements without a divine intelligence."

Archimedes' planetarium probably represents the highest point of Hellenistic Science, and today we can only have a vague idea of it by looking at the astronomical calculator of Antikythera, which was in all probability a portable version of the planetarium.

The siege of Syracuse is also sadly known because it was during this siege that Archimedes lost his life. Plutarch, in his *Life of Marcellus*, reports three different versions of the death of Archimedes: all versions agree in the fact that he died by the hand of a Roman soldier, although the Syracuse scientist is said to have been extremely appreciated by Marcellus, who seemed to be grieved by his death and gave him an honorable burial.

The death of Archimedes marks a symbolic point of no return for Hellenistic Science: after this event the phase of decline begins. As we will see, the decline is not immediately recognizable as such, but is usually preceded by a phase of mannerist fashion in which there are no more original ideas, but only repetitions and progressive refinements of pre-existing ideas. We believe that the siege of Syracuse and the consequent decline of Hellenistic Science started an inexorable process that, centuries later, will lead to the Dark Ages.

In the artistic domain, the Renaissance was followed by Mannerism, which in its most negative sense is depicted as the artistic current in which the artist no longer seeks inspiration in nature, but limits himself to attempting to imitate the works of the three great Renaissance artists, Leonardo, Michelangelo and Raphael (thus losing the instinct of originality that had characterized the Renaissance artist). Similarly, Roman art limited itself to copy and reproduce Hellenistic masterpieces. In the same way, and we could say cyclically, in every stage of history of Science one recognizes a phase of maximum development followed by a *mannerist* phase, which preludes to an imminent decline eventually followed by another growth stage: we believe to be followers of Giambattista Vico's doctrine. The great scientific advances of the Hellenistic period, which in the restricted field of Astronomy the available sources attribute mainly to Eudoxus, Aristarchus and Hipparchus, are followed by a phase of stagnation in which attention is focused on computational aspects and loses, therefore, that originality which had characterized the scientific revolution of the IV-III century BC.

In this mannerist framework stands Claudius Ptolemy (100 AD - 170 AD), whose greatest contribution to ancient Science consists in the refinement of the algorithm, originally due to Apollonius of Perga, that allowed to calculate precisely the positions of the planets of the Solar System. Ptolemy worked in Alexandria when probably the Library still existed and therefore had at his disposal the largest database in the world to which one could have access in that time. It is peculiar that, while Ptolemy was concentrated purely on the problem of calculating apparent motions of stars in a geocentric reference, his successors attributed to him the choice of the geocentric model of Eudoxus. We do not believe that Ptolemy had consciously refused the heliocentric model of Aristarchus: like many modern engineers he was only interested in practical calculations, and spent all his time in describing calculation algorithms. In any case, it is necessary to point out that, as we mentioned when discussing the Antikythera calculator, the calculation by deferents and epicycle had been introduced by Apollonius of Perga centuries before Ptolemy. From this consideration Ptolemy appears to be a compiler of already known results rather than the inventor of something new.

Ptolemy's algorithm turns out however to be a computational tool more precise than the algorithm developed by Apollonius of Perga and capable of giving estimates of the positions of the planets with sufficient precision for the astronomy of his time. We stress that his time is quite different from the centuries in which Eudoxus, Aristarchus and Hipparchus operated and in fact, for example, the estimate given by Hipparchus of precession is significantly more accurate than that made by Ptolemy three centuries later. The algorithm is based on a system of successive approximations made of compositions of uniform circular motions on circles of different sizes and with the centers located at *ad hoc* chosen points. An epicycle is a circumference whose center is placed on the circumference of a larger circle called deferent. In the model of Apollonius of Perga, therefore, the planetary orbits are represented as a composite motion of the revolution of the planet along the epicycle and of the epicycle along the deferent. By increasing the number of epicycles, one can obtain more and more accurate estimates of the orbits of the planets: one can conjecture that Apollonius of Perga was aware of the fact that increasing the number of epicycles one could reduce the error in the estimates of the kinematics of planets. We doubt that Ptolemy had this awareness. This multiplication of epicycles has been widely criticized in the past by the followers of Copernicus (1473-1543), against the opinion of some Jesuit erudites: the fact that more precise estimates could be obtained by increasing the number of epicycles was seen as an unnecessary complication of the model. In fact, the controversy between Copernicans and some Jesuits was based on a fundamental misunderstanding: while Copernicans considered the number of circumferences involved in the mathematical description of Solar System as a part of a postulation scheme, and therefore wanted to reduce it using Occam razor, their Jesuit opponents stressed the mathematical aspect of the question, remarking that periodic motions can be approximated better and better by increasing the number of epicycles.

As reported by Gallavotti [41], since Schiaparelli's analysis [38] the approximation technique via epicycles for the periodic motion of planets can be recognized as an initial form of Fourier analysis. As it is well known, Fourier (1768-1830) joined Napoleon's Egyptian campaign in 1798. The development of his analysis, conversely, dates from 1822. We have, in the present work, repeatedly conjectured, sometimes even demonstrated, that in the history of the transmission of scientific thought the often unmentioned source of works that are perceived as revolutionary and forerunners for modern Science is to be found in works of the Hellenistic age, which are nowadays (perhaps not by accident) lost.

It is clear that Fourier could have simply been inspired by what was already known about this technique. It is purely speculative to believe that he could have found other sources while campaigning in the place where the largest Library in the ancient world had risen. It is also clear that Apollonius' model and Ptolemy's algorithm were known, at the expense of Aristarchus' heliocentrism, throughout the Middle Ages and were considered basic until Kepler (1571-1630).

5.7 The materialization of Eudoxus' model

It is remarkable that during Dark Ages a choice among available models for the Universe was made. Soon one model was confused with reality. In fact, it was the simplest, and less predictive, model to be confused with reality for at least six centuries (the time interval between the fall of the Western Roman Empire, i.e. 476 AD, and the small Renaissance of Frederick II Hohenstaufen, who lived between 1194 and 1250). The complexity of all models formulated after Eudoxus was totally out of the understanding possibilities of nearly every intellectual of that Ages. Therefore, what could not be understood because of its sophistication and complexity was rejected as false, a useless and empty complex philosophy, soon associated with useless mathematics. Instead, naive and primitive models were promoted to crystal clear truths, that one could not dispute without risking to be considered heretical.

Moreover, the concept of mathematical description of phenomena and the role of mathematical entities used to predict them were completely lost and therefore, while describing Eudoxus' model, it was felt necessary to materialize the hinged rotating spheres assuming (as also it has been recalled explicitly by Schiaparelli) that the Universe was closed by crystal vaults mechanically interconnected one to the other.

One can get a clear idea of how much the thinking has regressed with respect to the Hellenistic period by considering that Bede the Venerable (672-735), one of the greatest scholars of the period immediately following the collapse of the Western Roman Empire, is remembered for having invented a method of counting up to a million with the fingers of the hands. So great is the devastation following the end of Hellenistic Science that mankind had to learn again how to count!

The lowest point in the scientific understanding during the Dark Ages occurs with the materialization of Eudoxus' model of homocentric spheres. Paradoxically, in opposition to the state of intellectual disruption produced by this materialization, from an artistic point of view the distorted view of Eudoxus' model generates a series of masterpieces in figurative art that perhaps had, at least, the merit of inspiring the efforts of Renaissance scholars to restart the systematic study of the problems addressed by Hellenistic scientists.

The materialization of the model of the homocentric spheres leads to two misunderstandings, the former of a purely scientific nature and the latter of a socio-cultural nature. The first misunderstanding concerns the vision of the universe that the man of the Middle Ages has: the Sun and the planets not only rotate around the Earth, but also they are stuck on metal rings (or crystal vaults) hinged to each other and rotating around the center of the Earth. This abnormal misunderstanding is generated by the literal interpretation that the majority of the medieval intellectuals were able to give of the drawings representing the homocentric spheres or of their practical realization in the ancient Greek armillary spheres (and in fact the armillary spheres begin to be spread again in Europe in the Late Middle Ages).

The second misunderstanding, as we said above, is of socio-cultural character and concerns the perception that the modern History of Science has of Eudoxus and his model. As we have repeatedly emphasized, relying also on the opinion of Schiaparelli, Eudoxus was fully aware that his model was not the physical reality and that, for example, the homocentric spheres represented only the elements of a mathematical model of reality and absolutely not the reality itself. Instead, partly because of medieval misunderstandings about it, the common perception of much of the modern scientific world is that Eudoxus, and Hellenistic scholars in general, had a very naive idea of reality and of its mathematical modeling. With due differences, it is as if in a thousand years from now our descendants will report that we were convinced that electrons are yellow balls with an arrow stuck along one of their diameters just because in some physics textbooks similar images are proposed to give an approximate idea of the spin. Of course, there are in present times some physicists who have such a belief: however, nobody attributes it to Wolfgang Pauli (1900-1958)! Similarly, we should respectfully appreciate Eudoxus' vision of Science.

We can attribute to two factors of fundamental importance the fact that after a thousand years of darkness the flowering of the Renaissance revived scientific interest. The first and inescapable factor consists in the fact that during all the Middle Ages the only scientific discipline that continued to be taught and transmitted from *maestro* to *pupil* was Euclidean Geometry. The presence of Euclidean Geometry in the cultural background of the first humanists certainly allowed them to appreciate the importance of the content of the ancient Greek texts of the Hellenistic school and to be able to read them. Obviously, not all humanists had the same skills and the same preparation and, for example, as discussed in detail in other chapters of this work, Tartaglia is not able to fully understand the reasoning of Archimedes and therefore modifies Archimedes' figures considering them wrong.

The second very important factor is given by the Byzantine cultural school, which, differently from the Western one, had remained active until the fall of the Byzantine Empire, which occurred in 1453 with the fall of Constantinople. One of the most important intellectuals of the 9th century Byzantium is Leo the Mathematician or the Geometer (790-869). This erudite had all the skills which will be found in the future humanist and, indeed, we due to him and to his farsightedness, probably, the first spark of Humanism and Renaissance in Europe. Leo the Mathematician commissioned the copy of many Hellenistic scientific manuscripts and, among the others, of the works of Archimedes. At least three manuscript containing the works of Archimedes were produced under his responsibility, today known as codices A, B and C. When Byzantium was sieged and conquered by the Crusaders in 1204, Leo's library was dismembered and a part of the manuscripts stored in it was brought to Europe. The presence of all these Hellenistic works in Europe gave rise to a strong revival of interest in science, and for the first time in a thousand years scientific progress started again. The Renaissance had begun.

6 The postulations of Mechanics, forces and their materializations

From now on we will focus on another materialization of mathematical concepts which is still occurring in many scientific milieux. While armillary spheres are not believed to be real anymore, there are too many contemporary scholars who managed to persuade themselves about the *reality* of forces. These scholars talk about forces as if they were objects one can observe in real world: forces seem to have, in somebody's words, the same ontological reality as walls, boats, wind, pulleys.

6.1 The recovery of ancient Hellenistic Mechanics: Middle Ages mechanicians

It has to be remarked that many historians talk about the "Renaissance of XII century" as a period in which in Western Europe the Latin translation of Greek and Arabic works, especially in Natural Science, Philosophy and Mathematics, greatly changed the cultural standing of Latin-speaking culture. In this period, and after it, we meet many erudites and scholars who tried to recover the lost Hellenistic knowledge.

While we have some ancient sources about the Mechanics of material points, we can only conjecture the existence of an Hellenistic Mechanics of deformable bodies. The most meaningful hint indicating its existence can be found in the works of Galileo Galilei [42]. Exactly as he tries to reconstruct Seleucus' theory of tides, and as he tries to reconstruct the theory of planetary system by Hipparchus, Galilei also tries to understand the theory of deformable beams: though, we must say, without great success [42]. In fact, Galilei did not manage to understand how bending stiffness of a beam depends on the geometry of the beam cross-section: his deduction starts from a wrong conjecture about the deformation field inside the section. By the way, the fact that there is an evidence that Leonardo da Vinci (1452-1519) tried to understand the theory of beam is another hint about the existence of an Hellenistic source in the subject, as Leonardo is known to be a great estimator of Greek Science.

The difficult point that Galileo did not manage to fully understand concerns the deduction of a theory of a 1-dimensional continuum (the simplest being Euler-Bernoulli beam theory) from a more detailed 3-dimensional continuum theory. In fact, the process of micro-macro identification has been fully developed only when the variational postulations of Mechanics have been recovered [43–47]. Galileo did not conjecture the right linear dependence of the contact force intensity on the distance from neutral axis in beam theory: clearly, it is extremely useful, if one wants to develop generalized beam theories, to understand how the progenitor theory has been formulated [14, 48–58].

The existence of more ancient (and sometimes partially lost) sources may contribute to explain the reasons why in Mechanical sciences one observes very often, especially when considering Middle Age texts, some oddities in the diachrony of Mechanics development. One observes more advanced texts which are precedent to less advanced ones and to definitely primitive others. The existence of linguistic and social barriers does not seem enough to explain the mentioned observed evidence: we believe that some scholars could access to sources that were very faithful to the original Hellenistic thought while others had access to worse sources or, even, could not understand really the content of such sources.

The strangeness in the diachrony of the development of Mechanics leads us to conjecture the following hypothesis: the distribution and accessibility of the ancient Hellenistic Science texts, often available in a single copy, modulate the speed and readiness of the rebirth of scientific thought. With the fall of the Western Roman Empire, the few remaining elements of unity in scientific thought collapsed, and the enormous progresses made in the 4th and 3rd centuries BC were relegated to the monasteries that owned the only medieval libraries. And in the monastery libraries these texts were sometimes lost and the genealogy of scholars, with a Maestro explaining to the pupils the content of the ancient texts, was broken. The practically null scientific preparation of medieval westerner scholars decreed the loss of many Greek scientific works. One who does not understand what she/he reads prefers to believe that what she/he is reading is, at least, useless and, therefore, unworthy of being transmitted to posterity. Thus, unlike works of literature, philosophy, historiography and other non-scientific disciplines, not only scientific texts were not copied, but often the very expensive parchment on which they were written was reused. This is precisely what unfortunately happened to codex C of Archimedes' works: the parchment was scraped off to make space for prayers against the flu.

It is not surprising that one of the greatest intellectuals of this period was Bede the Venerable (672-735)! The only significant aspect in the scientific sphere of this period is that the habit of studying Geometry remained almost intact and that Euclid's Elements remained one of the essential texts even during the Middle Ages: in fact, geometry was taught to all scholars. It seems that one of the few copies of the Elements of Euclid was preserved to posterity by the family Hohenstaufen and, in particular, by Frederick the Second. We can make the following conjecture: it was only thanks to the education in abstract thought provided by Geometry that a return to Science was possible in the Renaissance. Or, at least, this return required less time than it would have been necessary if Euclidean Geometry had not been taught during the Middle Ages.

Perhaps the clearest sign of the scientific regression of the Middle Ages is the theory of the nine medieval heavens, which is directly induced by the complete misunderstanding of the theory of the motion of the planets. At the beginning of the previous section we showed the sad fate of a proto-scientific theory, the one that was supposed to be at the basis of Nebra's disc: with the demise of the society that produced the theory and the birth of a non-scientific society, the scientific content of an abstract theory is completely lost and subsequently distorted into a religious belief.

After the *small Renaissance* of Frederick II Hohenstaufen there was a slow rediscovery of the distinction between model and real object: the basis of this slow process was the return to the study of Logic, which provided a structure for the subsequent return to scientific thought. An important role in this rediscovery process was played by William of of Occam (1285-1347), whose *Summa logicae* (c. 1323) constitutes a kind of meta-theory necessary to formulate theories. In fact, an important first step towards the rediscovery of scientific thought is represented by the so-called Occam's razor[6]:

Pluralitas non est ponenda sine necessitate[7]

It was only after 1100 AD that Latin translations of textbooks on Logic developed in Arab cultural circles started to arrive in Christian Europe. In Arab Science, in fact, we find the eminent scholar Avicenna (980-1037) that tried to propose a proto-inductivist system of Logic which is alternative to the Aristotelian one. Avicenna's Logic also influenced Western thought and we can say that, in a certain sense, Avicenna can be considered one of the founders of scientific inductivism, which will be discussed shortly below.

If we want to get an idea of how much the collapse of the Western Roman Empire influenced the collapse of scientific thought, we can consider the apparent anachronism observed in the cultural milieux of the capital of the Eastern Roman Empire, Byzantium (i.e. Costantinople), where the cultural ferment that had characterized Hellenistic circles survived for a few centuries. Byzantine intellectuals, whose works would only be rediscovered later in Western Europe, provided the first westerner humanists with a key for decoding Greek scientific thought, and , in particular, Greek Mechanics. John Philoponus (490-570), for example, proposed the concept of *impetus*, which seems strongly related to the concept of inertia, about a thousand years before Galileo and Newton. Moreover, Byzantium also represents a sort of *reduced Alexandria*, collecting the knowledge of the time and organizing it for later dissemination. Among the various examples of this Byzantine ferment, we cannot forget the already mentioned Leo the Mathematician, promoter of the renaissance of mathematical studies and of the rediscovery of Archimedes' scientific personality. Here, we will limit ourselves to recall that it is to him that we owe the rescue and transmission of many of Archimedes' texts, which only reached our hands thanks to the copies he commissioned. Several wars and an unfortunate Crusade that diverted Christians from the liberation of the Holy Sepulchre to the sack of Constantinople, which occurred in 1204, later, disseminated throughout Europe the famous Archimedean codices A, B and C. Albeit he was living several centuries before, Leo the Mathematician can be considered a true Renaissance man.

In fact, the role of the scientific ferment of Byzantium is crucial in the development of the Italian Renaissance, albeit it would still need almost five hundred years for westerner intellectuals to stop discussing about the sex of angels and to devote themselves to problems of a less elevated nature, perhaps, but certainly more useful to the progress of humanity. On the path to the Renaissance, we can at least mention some scholars who characterized the slow rediscovery of mature scientific thought. Thomas Bradwardine (*Doctor Profundus*, floruit 1330) distinguished kinematics from dynamics, introduced the concept of instantaneous velocity and discussed the law of falling bodies. Nicolas d'Oresme (c. 1320/1325 - 1382), institutor of the Dauphin of France, studied the Universe

[6]Some modern scholars misunderstood Occam razor spirit and believed that it was forbidding theories in which too many parameters appear: in this way they exclude, a priori, any possibility to model complex mechanical systems, as those studied, for instance, in [25,27,59–61].

[7]Plurality has not to be posed without necessity.

with mathematical methods: in particular he formulated a Galilean invariance principle and established the foundations of Analytical Geometry. Many others would deserve to be cited: however, we simply want to stress here that we know for sure that at least in one époque the main work of scholars consisted in reading ancient books whose content was perceived as very profound albeit resulting very obscure.

In fact, we believe to recognize in modern times the slow rediscovery of ancient theories. Albeit this rediscovery occurs in restricted disciplinary subgroups of scholars, the features of the sociological process seem to be the same. Of course, the fact that other contemporary groups had not lost the knowledge which is being recovered for sure helps in the rediscovery endeavor. We believe that in theoretical Continuum Mechanics the rediscovery of Lagrange-Piola postulation for generalized continua in the group of scholars following Truesdell orthodoxy has been held by the existence of Landau textbook in Theoretical Physics[8].

6.2 Fundamental concepts and frequent misconceptions in the field of Mechanics of materials

In the previous sections we dealt with the Mechanics of material points and its applications to the description of the planetary motion, from now on we will focus on the Mechanics of deformable bodies. However, we will base our analysis on available sources, which are much more modern. It will be clear that the same sociological phenomena involved in the transmission of knowledge observed in the transmission of the Mechanics of material points through the centuries occur also in the transmission of the Mechanics of deformable bodies. Our description of historical development of the Mechanics of deformable bodies starts with the works of Gabrio Piola (1794-1850) [2,3] and continues until contemporary times: all sources are fully available.

Mechanics of deformable bodies studies how the equilibrium shapes of bodies change because of their interactions with the external world. A given body is assumed to be constituted, in every of its material points, by a specific material. The current shape of a body is kinematically modeled, since the fundamental work by Lagrange (1736-1813) [11], by means of a placement function. Each material is mathematically modeled, in its range of elastic deformation, by the corresponding deformation energy density, depending objectively on the gradient of placement. Further constitutive functions and kinematical descriptors need to be introduced for modeling damage, plastic phenomena, etc.

In this context, it seems absolutely meaningless the expression *natural material*. One may argue, in fact, that human activity did modify everything in the world (think, as an example, about forests: almost all of them have their present shape as the result of a human design). From an Engineering point of view, we can only talk about materials that have a simple microstructure

[8]Richard Toupin admitted (personal communication) that, since his studies on Landau's lecture notes, he always believed that Mechanics had to be founded on variational principles, notwithstanding what advocated by Truesdellians.

(i.e. the more often used, up to now, in Engineering) and materials that have a complex microstructure. We do not share the primitivistic belief that natural is equivalent to simple, also because the definition of simplicity depends on the particular historical period.

Every material which exists is natural. Of course, we may ask ourselves if it is possible to find an existing material whose behavior can be described by certain constitutive functions. It is, therefore, meaningful to establish some *physical admissibility* criteria for *logically conceivable* constitutive functions. For instance, by introducing constitutive functions for a material which allows some deformative cycles that produce energy, one can get a mechanical system which contradicts the Principle of conservation of energy. Clearly, such constitutive functions would not be physically admissible. In [62], it is clearly stated that, unlikely what believed by Truesdellian school, there are no elastic materials which are not also hyperelastic. Truesdell wants to try that there are relationships between stress and strain, in first gradient materials, which do not derive from a principle of minimum of energy in a stable equilibrium configuration. He wants to prove that a postulation based on the laws of balance of forces and moments of forces is more general than a postulation based on the Principle of Virtual Work. This effort, as we will discuss later, is vain as Gabrio Piola has proven [2, 3, 59] that in every generalized theory of continua balance of forces and moments of forces are necessary conditions for the validity of the Principle of Virtual Work, while there are generalized continua (for instance, second gradient continua [20–23, 63–75]) for which the balance of forces and moments of forces *are not sufficient conditions* to ensure the validity of the Principle of Virtual Work. This principle seems to be the most fundamental one in Mechanics. Therefore, the important question "what is a natural material?" can have a simple answer: it is a material which may exist.

Therefore, the theory of metamaterials, if one wants to avoid ontological paradoxes, cannot be defined as the theory of those materials which are not natural, because otherwise we were dealing with non-existing materials. Another possibility is to define metamaterials as those materials whose mechanical behavior is "exotic". Now the obvious question arises: what is an exotic mechanical behavior? The answer could be: an exotic mechanical behavior is a behavior which has not been yet observed. Of course, what is exotic in a certain historical moment may become standard in another one. For instance, Lamé (1795-1870), Navier (1785-1836), Cauchy (1789-1857), Poisson (1781-1840), all considered a material with negative Poisson's ratio as very exotic, and some scholars of their époque did even believe that such a material was unphysical (see [42, 76, 77]) and could not exist. Instead, auxetic metamaterials do exist and play a relevant role in modern Engineering.

Also in the group of scholars in Mechanics, albeit this theory is the eldest one in mathematical physics, some epistemological misconceptions are rather common. The main among these misconceptions are:

i. confusing a mathematical model for a material with the physical material itself (the same ontological misconception occurred to Eudoxus' model);

ii. believing that particular assumptions accepted for describing particular phenomena are universally valid in every physical situation (the same extreme platonistic or inductivistic misconception occurred in the school of Truesdell, where it is believed that every existing material must be modeled by first gradient continua, whose properties have been induced with experiments);

iii. refusing the Principle of Occam Razor by constructing theories with a series of ad hoc assumptions guided by experience (naive inductivism);

iv. believing that, simply manipulating a lot of data without any postulated model, one can predict, maybe using large computers, the behavior of physical systems (a modernistic form of naive inductivism, by means of which many want to find new metamaterials by simply divining in a random way metamaterial microstructures).

Concerning the confusion of a mathematical model postulated for a material with the physical material itself (ontological misconception), we must say that this is an old misconception that is very often met in history of Science. The example about the models of planetary systems can be considered a prototypal social phenomenon of this kind, because entire groups of scholars fall in this mistake. Usually, we have heard in debates among experts of Continuum Mechanics the following *wrong* statement: *Second gradient materials do not exist because used materials in Engineering do not show their properties and standard theoretical framework does not forecast them.* In this statement, one can find many layers of misunderstanding based on the following misconceptions:

(a) confusing first gradient continuum model (a mathematical model) with existing materials in nature (a physical object);

(b) believing that the standard theoretical framework, which has been paradigmatic in a school of Mechanics, includes every conceivable phenomena (this misunderstanding is induced by naive inductivism);

(c) confusing the standard first gradient continuum model with all used materials in Engineering (that includes both presently used and all usable in future physical objects);

(d) believing that, without having a theory describing it, one cannot use a material even when such a material is in her/his hands , with the paradoxical consequence that the material would not exist.

6.3 Mathematics designs the world: metamaterials, a change of paradigm

The mathematical modeling of physical phenomena has shaped the world, notwithstanding what *practical people* may believe. In fact, in Engineering Sciences the following phenomenon occurs: a theory is formulated, it applies to a specific set of physical objects and physical situations, therefore in Engineering practice only these objects and situations are considered for Engineering artifacts. The fact that one does not have a model describing the behavior a physical system, or some physical situations where a physical system can occur, or can be observed, implies that physical systems and situations which are not described, have to be carefully avoided in Engineering applications. For instance, if one has the capacity of calculating the deformed shape of a body only when linearized equations apply, then she/he limits the functioning regime of the artifacts which are built according to the above-mentioned theory to small loads, small deformations and small displacements.

Many engineers declare as a consequence that non-linear phenomena are not of interest in Engineering, with a typical process of removal of the complexity. In Engineering, non-linear phenomena are important; however, when they could not be fully studied with available mathematical tools, then they are avoided.

Therefore, the limits of our mathematical capacities limit consequently our predictive capacity and then our design capacity. For instance, sky-scrapers could not be built until Structural Mechanics became sophisticated enough to be able to design them. What can be mathematically conceived by means of a model can be transformed into an Engineering artifact, while every data-driven series of subsequent trials never produced any functioning Engineering solution. Data-driven research has produced, maybe, some interesting technical software solutions: however, when not guided by a clear modeling vision, it could not predict novel phenomena and seems to be a modern version of naive inductivism. On the other hand, in general, Engineering Sciences choose among available and conceivable systems those which can be mathematically modeled and limit its designing efforts to those for which mathematical predictions are possible, given the available computing tools. In few situations, on the contrary, Engineering Sciences attack a very difficult problem: that happens, for instance, when, given a mathematical model, the goal is to find a physical system which can be carefully described by that model.

On the basis of what we have discussed so far, it should be clear that a good theory is useless without suitable computational tools. This concept, which may perhaps seem trivial, assumes considerable importance if contextualized in the historical perspective that we have presented in the previous sections. The epistemological appreciation of the quality of a theory cannot prescind from the availability of suitable computational tools that allow for its use in getting predictions. A very detailed theory that cannot produce quantitative predictions is useless. If one theoretically tries to take into account too many phenomena, without considering the technical and computing difficulties which are found when applying such a detailed theory, then he/she does not supply the Engineering practice with a useful tool: being potentially capable to predict *everything* leads to the incapacity to predict anything. The classical example is given by the efforts of Navier [42] to develop a theory for predicting the deformation of a beam by starting from a *molecular* model: such a detailed model could not produce any prediction, due to calculation difficulties. Therefore, Navier was obliged to homogenize his discrete equations, for obtaining a computable model: his averaging hypotheses led him to believe that Poisson's ratio for isotropic materials could only assume the value 0.3, which is clearly against evidence.

Not only simplicity in the involved computing process must be required to a modeling effort, but also conceptual simplicity in the model formulation, that implies dramatic simplifications in the prediction process. Consider, as an example for this last statement, the relationship between the predictive capacity of Eudoxus' model of homocentric spheres and that of Aristarchus' heliocentric model: as we have seen, Eudoxus was unable to explain correctly, even by greatly increasing the number of spheres associated with a given planet, its retrograde motion, while it could be explained, instead, extremely clearly by Aristarchus.

Quantitative predictive capacity is inescapable in mathematical modeling and in its applications to Engineering Sciences [47, 55, 56, 78–83]: in fact, there is no scientific designing without accurate quantitative prediction. Furthermore, from a technological point of view, it is certainly easier to build a mechanism for getting predictions by using a model where all planets travel more regular orbits around the Sun, than a model in which the planets move seemingly randomly in the sky, traveling very irregular orbits, although the latter model, being geocentric, seems more faithful to observational reality.

In general, a model producing some *theoretically correct* or *physically intuitive* equations that cannot be efficiently solved is technologically (and scientifically!) of little significance.

A similar example is provided by what happens much later, when the Copernican system replaced the Ptolemaic system. Behind the process of substituting one model for another there is a technical consideration: predictions are obtained via computations and model development is constrained by available computational tools. The Copernican system did not give much more accurate predictions than Ptolemaic system. In fact, by adding a suitable number of epicycles, as rigorously proven by Gallavotti [41], one can approximate the apparent motion of planets as seen from the Earth as accurately as possible. Moreover, the kinematics of both systems are based similarly on the principle of the composition of circular motions. But the Copernican system is enormously simpler conceptually and allows for less laborious calculations, as Cicero did observe when describing Archimedes' planetarium mechanism. Technological capacities, in a sense, introduce a hierarchical ordering in the set of models: those models, for which simpler computing methods are available, become preferable.

We believe that both Eudoxus and Aristarchus did have falsificationist points of view when they formulated their models and that the debate inside Hellenistic Science about their competing models did involve only the models predictivity capacity. It was only after the decline of Hellenistic Science that models started to be confused with reality and that *true* models were opposed to *false* models: the loss of epistemological consciousness led scientists towards a vain search for ultimate truth. Therefore, after many centuries in which scholars were looking for ultimate truth and believed that such truth could be attained, the contraposition between geocentric and heliocentric models developed the characteristics of a religion war. Instead of debating about the predictive capacity of one model as compared with the other, the scholarly debate was involved in scientifically irrelevant questions concerning the role of religion, ethics and vision of life in Science. On the contrary, we believe that the true debate between scholars was not about the ultimate truth of geocentrism or heliocentrism but about the real aim of scientific research: is Science looking for ultimate truth (assuming that such a truth can be established once forever) or is Science formulating one after the other a series of conjectures to be tested with experimental evidence and possibly changed when such evidence requires it? Paradoxically, Cardinal Bellarmino (1542-1621), whose intention was to reaffirm that only theology had the capacity to reach ultimate truths, following the orthodoxy of St. Thomas Aquinas (1225-1274) and St. Augustine of Hippo (354-430), tried to get from Galileo a simple falsificationist statement about heliocentrism, while Galileo remained in an inductivist position, albeit formally changing his position in order to avoid to be condemned to be burned at the stake.

In a sense, a contemporary version of the debate involving geocentrism and heliocentrism is represented by the debate between the supporters of Cauchy postulation and d'Alembert postulation for the foundations of Mechanics. The Truesdellian supporters of Cauchy postulation believe that it is an ultimate and experimentally proven truth, which cannot but be improved by adding some epicycles, i.e. small corrections. Their attitude blocked the growth of generalized models in Continuum Mechanics. Another important circumstance to be taken into account, both when describing the paradigmatic change between geocentric and heliocentric models or between Cauchy continua and Generalized continua, is the development and improvement of computing tools that occurred during the change. While in Hellenistic times the only computing tools were based on a geometric understanding of the concept of real numbers, so producing mechanical computing devices like the Antikythera mechanism or the Archimedean planetarium, after the Renaissance of Science, first Copernicus rediscovered ancient heliocentrism. Subsequently Kepler could exploit the method of calculations based on Napier (1550-1617) tables of logarithms and finally Newton, by using Cartesian geometry, could get a prediction of the planetary motions without computing mechanisms. Therefore, it seems that, while being initially blocked in a inductivist epistemological view point, modern Science could improve its understanding of the planetary phenomenology, when compared with Hellenistic Science, only because the development of modern computation tools, based on algebra.

Coming back to Continuum Mechanics, we limit ourselves, here, to describe some fundamental points in the process that led to the introduction of Generalized Continuum Mechanics [16, 17, 84–92].

As we will see, Gabrio Piola introduced in 1848 the generalized continuum model based on the use of deformation energies depending on the $n-$th gradients of displacement, being aware of the conjectural nature of such mathematical models [33]. However, Cauchy and his followers did try to formulate the ultimate continuum model, based on induced true properties of matter, at macroscopic level. It is paradigmatic, in this context, the unconditional acceptance by Cauchy and his followers of the so-called *Cauchy postulate*, stating that contact forces, inside continua, can be only forces per unit area which, moreover, depend only on the normal to Cauchy cuts. Albeit Piola was well aware of the limits of this conjecture, whose applicability is limited only to a particular class of materials, and albeit Piola himself wrote clearly that Cauchy postulate had to be regarded as a constitutive equation, in a large group of scholars Cauchy postulate has been accepted as a religious ultimate truth that cannot be doubted. It is remarkable that Gurtin, who had started from an orthodox Cauchy-Truesdellian viewpoint, in his subsequent papers [93, 94] changed his fundamental postulation approach and, albeit ignoring Lagrange, attributes, with a typical modernist attitude, to an explicit Lagrangian follower (i.e. Toupin) the choice of what seems to be the most appropriate postulation of Mechanics. Piola's works were reappreciated only at the end of the 20th century, while his models had been rediscovered already 50 years before and had became the object of in-depth study in view of their potential technological application [20–24, 63, 64]. What has changed in the century and half that separates Piola's pioneering work from his (slow) rediscovery? Why did the Continuum Mechanics of the Cauchy school ignore (and in part still tries to ignore) Piola's results for over a century?

In Cauchy's version of Continuum Mechanics a number of *ad hoc* limitations are inserted, including the fact that the deformation energy of a continuum medium can only depend objectively on the first gradient of the displacement field. A priori, nothing would limit a dependence on higher order gradients, but the simplest choice, consistent with the phenomenology disclosed by Cauchy continuum model, is to limit oneself to the first gradient of the displacement. Piola, as we have said, introduces, for a purely logical need, the higher displacement gradients in the calculation of the deformation energy, and argues to characterize those microstructures for which homogenized models must be of this more general kind. Unfortunately, the differential geometric tools available to Piola did not allow him to characterize internal contact forces in the case of second and higher gradient continua: instead, he did manage to do so in the case of first gradient continua. It is not a coincidence that exactly when Piola succeeded in finding a representation for contact forces in first gradient continua, Cauchy (who probably met Piola in Italy during his exile following French July Revolution of 1830) developed his postulation scheme based on balance of forces and balance of moment of forces. It is only after more than a century that Paul Germain showed, in his fundamental work [33], which is the structure of contact forces in second gradient continua, by remarking that so-called Cauchy postulate is not valid for these continua and that edge contact forces may arise (see also [71, 72, 95]). Moreover, in [43, 96–98] it is proven that models where the second gradient of displacement acquires a non-negligible role, at macroscopic level, are obtained by homogenization starting from a microstructure, or architecture, at a lower scale in a continuum medium where high contrasts of stiffnesses are present. We believe that Piola had guessed this result: see [2, 3]. Therefore, in order to become able to evaluate and reveal experimentally the effects of the presence of the second gradient of the displacement field, it is necessary to have a technology which is capable of producing a microstructured material [43, 44, 99–103] and, above all, a material whose microstructure shows the suitable highly contrasted stiffness fields, so that, at the macroscopic level, the terms used by Piola and Germain in the deformation energy do appear (examples in which the required technology has successfully produced such microstructured materials can be found in [46,104–106]). This is a very clear example of how the limited technological capacity of an epoch can indeed block also its scientific development. As long as the lack of technological capacity does not reach a level where the results introduced in the new theories can be tested, the new theories will remain blocked, ignored and, definitely, unusable. The absurdity of contemporary situation lies in the fact that despite the technological ability to produce materials whose behavior is described by Piola's theory (and cannot be described in the framework of Cauchy models), there are still scholars who are obstinate in denying its usefulness.

The mathematical challenge that researchers in the field of Continuum Mechanics face today, therefore, is to design metamaterials that can be described within the framework of a generalized theory [86,87,90,107–109]. These materials, as we shall see, are conceived in order to possess mechanical properties that are significantly more performing than those of the commonly called *natural materials*.

Therefore, the fundamental problem in modern theory of metamaterials consists in the problem of synthesis of microstructures producing a specific desired macro-behavior [45, 69, 96, 110–112]. In fact, the modern challenge in the theory of metamaterials consists in finding that microstructure, or that micro-architecture, which, at the macroscopic level, produces a specifically required mechanical behavior. In this context, the most difficult problem to face from a mathematical point of view is to connect micro-structures and macro-behaviors. So, given a macroscopic theory (appropriate action functionals and consequent stationary conditions) one wants to find an algorithm to calculate the microstructure that, once homogenized, at the macroscopic level is described by the chosen macroscopic model [28, 113–118]. In this context it is important to remark that a major change in the research tools has been induced by the use of powerful computers to find suitable motions for minimizing postulated action functionals: in fact, especially in non-linear regimes, it is simply inconceivable to find closed form solutions and therefore only by means of suitably conceived algorithms it is possible to design and to predict the behavior of novel metamaterials [80, 119–124].

The basic ideas in the field of the synthesis of metamaterials may be borrowed from the *ancient* theory of synthesis of analog circuits. In this theory, it was possible to prove that every passive linear n-port element is algorithmically synthesized using inductors, capacitors, resistors and transformers [125–127]. In fact, it has been proven that, given the desired passive impedance, one can build a graph and can find for each branch of the graph a suitable circuital element such that the resulting discrete circuit has the chosen impedance.

The classical theory by Kron and McNeal [125,128] allows us, given quadratic Lagrangian and Rayleigh's potentials, to algorithmically determine the graph structure of the searched electric circuit and its elements *synthesizing* the given linear n-port element. This latter is mathematically characterized by its Lagrangian and Rayleigh's potentials. Therefore, at least in the theory of circuits, by starting from a finite number of basic microstructures and reproducing them at different length-scales, it is possible to devise a most general microstructure. The big challenge, now, consists in conjecturing that this method can be also applied to the synthesis of non-linear mechanical (and multiphysics) systems [126]. Some papers [129–132] can be referred to for well understanding the fundamental role of the synthesis process in metamaterial theory.

It has to be remarked that the theory of analog circuits has been partially lost and generally forgotten. The reason is that in the early '60s digital computing methods became *dominant* and analog computers were considered obsolete. As a consequence, it is becoming more and more difficult to find the sources in the theory of synthesis of analog circuits and a sudden change of paradigm occurred also in the textbooks of Mechanics and, in particular, of Structural Mechanics. In fact, many textbooks in Structural Mechanics in the '50s were full of schemes of analog circuits, considered very important for *practical applications*. One could have believed, by consulting such textbooks, that a structural engineer could not become a skilled professional without knowing the theory of circuits. It is ironic that after few decades the great majority of civil engineers do ignore even the existence of inductors and capacitors, not to mention transformers. This sociological phenomenon, that occurred in an époque when books are not easily lost and when a large number of scholars are active, proves three important theses:

i. loss of knowledge is a sociological process, which is always active in every group of scholars and in every society;

ii. in Science every knowledge may be useful in every other research field;

iii. there is no such thing as *obsolete* knowledge!

6.4 The Principle of Virtual Work and its correct application produces (generalized) Continuum Mechanics

Let us now proceed to examine in greater detail the distortion of sources and modification of basic principles that occurred in Continuum Mechanics. Unlike the conjectural study we have made of the development of planetary models in Hellenistic Science, in the case of Continuum Mechanics all modern (since d'Alembert *Traité de dynamique*, 1743 [133]) sources are available and therefore the reconstruction that we present here is not conjectural. However, in the evaluation of modern sources, we still have to consider a problem that may be considered logically absurd, and that yet, unfortunately, is having, also now, a considerable weight in the development of Continuum Mechanics: some fundamental sources in this field are not written in English (e.g. they are written instead in French and Italian) and, as a consequence, some scholars believe to be allowed to ignore them. This point can be fully developed when one details the study of Gabrio Piola's contribution to Continuum Mechanics.

Continuum Mechanics has been based by d'Alembert on the Principle of Virtual Work only. This principle allows for the calculation of the equations of equilibrium of a continuum and it is easily connected to the Principle of the Minimum Potential Energy for a stable equilibrium. As we will see, starting from Cauchy, Navier and Poisson, a very strong current of thought has been developed which has, in fact, replaced this fundamental principle with the independent postulation of the balance of forces and of the moments of forces, introducing some auxiliary concepts such as forces and attributing to them a fundamental role in Mechanics and in the phenomenology that it aims to describe. One must, however, agree on the fact that, especially when the physical system under examination is very complex, the Principle of Virtual Work is not only easier to apply, but it is also applicable when the balances of forces and moments of forces are not sufficient to characterize equilibrium.

We want to stress, in this context, that it is not by chance that mechanical systems, of interest in Engineering Sciences, were first studied through the application of this principle, introduced by Archytas of Tarentum in his *Mechanica Problemata*. While we do not know how Archytas had formulated the Principle of Virtual Work, it is evident that in his *opus* he uses it to study problems of applicative relevance such as the functioning of machines and levers (which are sometimes still studied in middle schools based on this principle, even before the concept of force itself is introduced).

An interesting problem related to the *Mechanica Problemata* is given by the following question: should it be considered as an exercise book whose reading had to be combined with a more theoretical work? Could the theoretical work have been lost? To give a definite answer to this question is not possible: however, we can make some conjectures, by considering the analogy with other pre-Hellenistic authors. It is, in fact, now widely accepted that, for instance, the production of Plato (c. 428/427-348/347) was of a twofold nature: one part of the works, those dedicated to his pupils, was of an extremely technical nature and specific for *experts*, with a level of complexity equal to the surviving works of Aristotle; a second part of Plato's works was of a rather popular nature. The latter were written in the form of dialogues and were thus more accessible to the general public. For what concerns Plato, the most technically difficult works were lost and only the popularizing ones were transmitted to us, while the opposite happened to Aristotle (384-322 BC). We remark here that one finds an enormous body of critical works commenting philosophical, historical and literary ancient production, while such an analysis is not dedicated to ancient scientific texts, so that the reasons for this kind of selection of transmitted scientific works were not deeply investigated. In any case, it is a reasonable conjecture to assume that Archytas may have written a simplified and applied version (i.e. the *Mechanica Problemata*) of a more complex work, in which, for instance, the Principle of Virtual Work was formulated in a more explicit way. If this conjecture is true (remember that there are still many texts of Hellenistic Science that are preserved in libraries and remain forgotten because today it is rare to find a scholar who knows mathematics, Greek, Armenian, etc.), it would imply that the presumably lost text of Archytas was extremely more abstract than the *Mechanica Problemata* and, therefore, that constituted a masterpiece in the field of Mechanics. Its importance could consist in the clarification of the mental process that led the first scholars in Mechanics to find its conceptual bases.

However, we cannot exclude the other possibility to be considered about Archytas' text: that it was an autonomous and self-contained work. This would support a different hypothesis of historical and epistemological relevance: perhaps, in the early phases of Mechanics, theory and exercises were mixed up in a single treatise. Perhaps the approach to complex problems was deliberately simplified by the proposal of a series of applied examples. This approach is the one preferred by some modern textbooks in Physics [134], based on the idea that a student will understand general concepts by inducing them on the basis of many examples. This approach is, instead, considered not efficient by those who study Mechanics from a deductive postulation point of view [20–22, 24, 63–66, 68]. The reader will understand that knowing how the Principle of Virtual Work was first formulated could be very important to settle this controversy. In any case, the text of the *Mechanica Problemata* that has come down to us is already rather abstract. In some places it seems to refer to concepts already known to the reader, just as one often reads in a modern text of solved exercises! For this reason we believe that the possibility of a second work that dealt with a complete and rigorous treatment of the theory behind the practical examples cannot be excluded.

We now want to discuss, in an obviously simplified and concise way, some aspects of fundamental importance in the formulation of the Principle of Virtual Work. A first aspect to underline lies in the fact that, as we also mentioned before, it is well-known and universally accepted, at least since the works of Archimedes, that to a stable equilibrium configuration corresponds a minimum of the total energy (Total Potential Energy Minimum Principle). It can be also demonstrated, using some mathematical reasonings, that the Total Potential Energy Minimum Principle implies a stationary condition (the first variation of Total Potential Energy is zero in its minima) that, on its turn, can be regarded as a particular form of the Principle of Virtual Work. Therefore, the validity of this form of the Principle of Virtual Work can be deduced as a consequence of the Total Potential Energy Minimum Principle. The Total Energy Minimum Principle can be formulated as follows:

> **Total Potential Energy Minimum Principle:** *The stable equilibrium configurations are the only ones for which the total potential energy has a local minimum.*

A necessary condition for stable equilibrium can be formulated if the total potential energy is differentiable with respect to the variation of configuration.

> **Necessary condition for equilibrium:** *Starting from a stable equilibrium configuration, the first variation of the total energy corresponding to each virtual displacement is zero.*

A virtual displacement is simply a small variation (more precisely, an *infinitesimal* one) of the body's configuration (but respectful of internal constraints and of kinematic boundary conditions) to be added to the tentative minimum energy configuration in the verification process aiming that such tentative minimum energy configuration is effectively of equilibrium. This formulation requires the deduction of a number of non-trivial mathematical results, which make the treatment of Mechanics by means of variational principles complex, whose esoteric content is reserved to scholars having a deep knowledge of complex mathematical theories. It is very presumable that this is the fundamental reason why some scholars decide to ignore completely this approach to Mechanics and turn, instead, to the simpler, but somehow incomplete (and surely unfit for the discovery of novel models) formulation based on the postulation of balance of forces and moments of forces. To give an idea of the mathematical difficulties implied by the postulation of Mechanics based on variational principles [45, 61, 120, 135–137], one must think that there is a whole branch of mathematics, the Calculus of Variations, which was developed to supply the needed conceptual tools to Mechanicians. It is suggestive to think that Calculus of Variations has deep roots in Hellenistic Science, as witnessed by the fact that isoperimetric problems are traditionally called also *Dido's problems*. As it is reported by [138], our conjecture is not too much daring. In fact, in the *Synagoge* by Pappus of Alexandria (c. 290 - c. 350 AD), as well as in the commentary by Theon of Alexandria (c. 335 - c. 405 AD) on Ptolemy, which both were transmitted to us, Zenodorus (c. 200 - c 140 BC) treated isoperimetric plane problems in a treatise which was lost. It is remarkable that Dido's problem was formulated and solved by Zenodorus, albeit we do not now the methods that he had used.

We can make a list of the main mathematical difficulties to be faced when deciding to resort to a formulation of Mechanics based on the Total Potential Energy Minimum Principle; indeed to this aim it is necessary:

i. to introduce the concept of infinitesimal variation of a configuration (otherwise called *small displacement*);

ii. to introduce the concept of work done by an interaction on a virtual displacement and the concept of virtual displacement itself;

iii. to define the first variation of a functional in terms of Taylor series developments (which, despite the simplicity and elegance of this powerful mathematical tool, appears to be indigestible to many scholars).

As mentioned above, the Total Potential Energy Minimum Principle implies the more general Principle of the Virtual Work, which can be also formulated in a simpler way from a mathematical point of view. Probably, in order to be able to understand in detail the efficacy of the variational approach to Mechanics, and, at the same time, in order not to be discouraged by the mentioned difficulties, it can be useful to refer directly to the formulation of the Principle of the Virtual Work given by d'Alembert, who was the first, in the modern age, to found

Mechanics on it. In fact, in his treatise of 1743, d'Alembert formulated the Principle of Virtual Work in a more modern language with respect to that found in Archytas' *Mechanica Problemata*. d'Alembert's formulation generalized the previous formulation of stationary condition implied by Total Potential Energy Minimum Principle, and, because of its greater generality, it allows for a better focus on the key points of the variational approach:

> **Principle of Virtual Work (d'Alembert, 1743):** *A system is in equilibrium in a given configuration when the total work done by all interactions involving the system is zero for each virtual displacement from that configuration.*

From a correct application of the Principle of Virtual Work, one can obtain the equations of equilibrium of a mechanical system, also known as its Euler-Lagrange equations. It is to Lagrange that we owe the application of this principle to a wider class of mechanical systems. In the last version of his *Mécanique Analytique*, Lagrange formulated the Principle of Virtual Work for a continuum system and applied it to the study of the motion of fluids. In his nomenclature, Lagrange called *power* what will later be called force, and *momentum* what we know today as power. He claimed to prefer this nomenclature as it had been previously chosen by Galileo Galilei: we agree with his motivations, and we regret that unfortunately his suggestion has not been accepted in mechanical literature. Some, rather naively, from this different nomenclature used for the mathematical objects used by Lagrange, *deduce* that Lagrange did not understand the problem he wass formulating. Once again, we observe this modernist attitude that wants to judge the past by current conventions. Quoting Shakespeare[9]:

"*What's in a name? That which we call a rose*
By any other name would smell as sweet".

To show how the ideas of d'Alembert were elaborated and improved by Lagrange, it is very useful, finally, to introduce the formulation given by Lagrange[10]:

> **Principle of Virtual Velocities (Lagrange):** *If a system constituted by bodies or points, each of which is pushed by any power, is in equilibrium and if a small movement is given to this system, by virtue of which each point will cover an infinitesimally small distance that will express its virtual velocity, then the sum of the powers multiplied by the distance covered by the points where it is applied along the line of application of this same power will be equal to zero, if we consider as positive the small distances covered in the same direction of the power and as negative the distances covered in the opposite direction.*

[9]W. Shakespeare, Romeo and Juliet, Act II, Scene I.

[10]The translation has been performed by the authors of this Chapter from the Lagrange's original text.

Although a modern formulation of this principle usually includes the use of concepts from functional analysis, tensor algebra and mathematical analysis, one has to agree on these points:

i. Lagrange's formulation seems so general that it includes all the versions that have been formulated so far;

ii. this formulation uses the minimum possible mathematical concepts (i.e. only concepts from Euclidean geometry) that are sufficient to rigorously express the principle in its full generality.

We can, therefore, conclude that it has been correct to call, in the past, mathematical physicists with the attribute of *Geometricians*, as it was the geometrical language that allowed for the first formulation of mechanical theories.

The life-long work by Piola consisted in completing the work that Lagrange had left to be completed after his death. Piola, also formulating a micro-macro identification procedure, in 1848 published a fundamental work [33] where:

i. he deduced a macroscopic model describing the overall behavior of a system of a large number of interacting particles, obtaining also macroscopic constitutive equations in terms of microscopic geometric and mechanical properties;

ii. he introduced nth-gradient continua, determining the conditions for which they must be used in order to describe correctly the behavior of microscopically complex mechanical systems;

iii. he determined the structure of contact forces for first gradient continua, as studied by Cauchy, and discussed in his own work of 1822.

It has to be remarked that until 2012 [95] the determination of the correct form for contact interactions in nth-gradient continua was not obtained. Piola did not have at his disposal the mathematical tools from differential geometry, that were developed also, together with Gauss and Riemann, by the Italian school of Piola's scientific lineage, i.e. Francesco Brioschi (1824-1897), Eugenio Beltrami (1835-1900), Gregorio Ricci Curbastro (1853-1925) and Tullio Levi-Civita (1873-1941). However, Piola could prove, for a generic nth-gradient continuum, the following theorem, starting from the Principle of Virtual Velocities:

> **Balance of forces and moment of forces (Piola):** *if a deformable nth-gradient continuum body is in an equilibrium configuration, then the resultant and resultant moment of applied external forces vanish.*

Piola, following d'Alembert and Lagrange, defines resultant forces and resultant moment of forces as the vectors needed to represent the work of a system of forces in a rigid virtual motion. Therefore, these concepts are mathematical abstract constructions that can be used to calculate equilibrium configurations. Resultant forces and resultant moment of forces are mathematically defined in order to allow for the characterization of equilibrium configurations and do not correspond to any directly measurable physical quantity. Remark that, while for first gradient continua Cauchy has proven that this necessary condition is also sufficient, in general, for a subclass of second gradient continua [23] and for all nth-gradient continua with $n \geq 3$ balance of forces and moment of forces select a set of configurations greater than the set of equilibrium configurations. In fact, in higher order continua contact interactions are not limited to forces and couples.

One can assume that the equilibrium necessary conditions, given by resultant forces balance and resultant moment of forces balance, represent all Euler-Lagrange conditions of the Total Potential Energy functional for first-gradient continua only, and as such, in this case, directly provide the governing equations in *strong* form; it is clear, however, that in a numerical approximation setting it is always convenient to reformulate such conditions in *weak* form, and for this purpose the use of a variational principle is preferable because it lends itself directly to providing the governing equations in an easily discretizable form.

6.5 Confusing a necessary condition with the fundamental Principle: the materialization of forces, i.e. auxiliary mathematical concepts

As we have seen in the previous section, forces, and, in particular, contact forces, are a mathematical artifice introduced in order to deduce some consequences of the fundamental postulate of Mechanics, the Principle of Virtual Work, and, as such, they are of no use outside this context. Contact forces are, then, a mathematical invention, developed in the centuries to find some logical consequences of the above-mentioned Principle. The genesis of the concept of force and all the misunderstandings to which it was subject deserve an in-depth analysis, which is beyond the scope of this work (but see [139]). It is remarkable, however, that Archimedes did introduce in his *On the floating bodies* the concept of pressure and that the first textbooks in modern Mechanics (those, already cited, by d'Alembert and Lagrange) did apply the Principle of Virtual Work to deduce the equations of equilibrium and motion of perfect fluids.

The main idea that leads to the definition of resultant forces and resultant moment of forces can be traced back, in modern Continuum Mechanics literature, at least to Gabrio Piola. It is unfortunate that the complete works by Euler had not been published in an English translation until the second half of the twentieth century. The enormous corpus of the works by Euler, all written in Latin, may include some applications of the Principle of Virtual Work, or of the Total Potential Energy Minimum Principle, leading to the definition of resultant forces and resultant moments of forces, as the tradition in Mechanical literature attributes to Euler the introduction of these necessary conditions for equilibrium. We could not find any textbook clarifying this point and the original works by Euler are not easily accessible: however, see [42], Euler did deduce the equations of Elastica by using the Total Potential Energy Minimum

Principle. It is remarkable that Truesdell wrote more than 400 pages in the series of Springer volumes gathering Euler Opus, without translating a single word of Euler's text.[11]

In order to characterize equilibrium configurations by using the Principle of Virtual Work, one can consider for every involved body those rigid virtual displacements that are allowed by applied constraints. In absence of applied constraints, therefore when we have a free body, the work done by externally applied loads on rigid displacements can be represented as linear functionals on the pair of vectors composed by translation and rotation velocities. By Riesz (1880-1956) representation theorem, these linear functionals are uniquely determined by two vectors when using inner product for calculating the represented functionals images. The vector whose inner product with translation virtual velocity gives the virtual work done is called *resultant force* of the applied loads, while the corresponding vector giving the virtual work in correspondence with virtual angular velocity is called *resultant moment of forces* of the applied loads. It is therefore clear that the concept of force is generated while developing a mathematical theory to be used for deducing logical consequences from the basic principles of Mechanics. Albeit it is of great importance, it does not correspond to any directly measurable physical quantity and is a pure *superstructure* of use in mathematical reasonings: exactly as it happened for Eudoxus' spheres or Apollonius' epicycles. When a free body is in equilibrium, then the total work done on rigid displacements from equilibrium configurations must vanish, and, as a consequence, the resultant force and resultant moment of forces must vanish. It has to be remarked that also when the body is deformable such *necessary* conditions must be verified, whatever may be said by some scholars of Truesdellian orthodoxy. Moreover, by introducing Lagrange multipliers, one can add to active forces also reactive forces, in presence of constraints. Including reacting forces in the set of applied external loads, one gets the validity of fundamental balance equations for Mechanics (i.e. resultant forces equal to zero and resultant moment of forces equal to zero) also in the case of deformable constrained bodies.

When trying to calculate equilibrium configurations using analytical methods, fundamental balance equations are the most useful tool to be used in calculations. However, when using numerical computational tools one must resort either to the Total Potential Energy Minimum Principle or to the Principle of Virtual Work.

The most recent materialization of abstract mathematical concept can be observed when, within the framework of Truesdell's presentation of Continuum Mechanics, one finds statements attributing to the concept of forces a physical reality and when one reads that *the laws of balance of forces and moments of forces are based on physical evidence*. It is as if one could measure a functional defined in a Sobolev space and could get information about it based on physical intuitions.

[11] Leonhardi Euleri Opera Omnia: Opera mechanica et astronomica. The rational mechanics of flexible or elastic bodies, 1638-1788 : introduction to Leonhardi Euleri opera omnia, vol. X et XI seriei secundae / C. Truesdell, Volume 10. Springer, Zurich, 1960.

Just as Eudoxus' model made sense and clearly served a certain purpose even though it had no pretension of being a *pictorial* description of physical reality, so the concept of force has a very precise reason for existing in the context of the theory where it was formulated: d'Alembert, Lagrange, Piola used this purely mathematical object to formulate a theorem by means of which the equilibrium equations of a mechanical system could be derived. The aim of this formulation was in fact to obtain a way of characterizing equilibrium, similarly as Eudoxus' aim was to find a way to describe the motions of the planets, including retrograde motions. When, however, the scientific epistemological consciousness decays, then there are those extemporary scholars who fail to understand the difference between models and reality, and, in the total resulting confusion, it happens that objects of secondary importance, such as the homocentric spheres for Eudoxus or the forces for Continuum Mechanics, become preponderant.

The confusion becomes total when Cauchy tries to deduce the equilibrium conditions in Continuum Mechanics by postulating the existence of the stress vector, in order to calculate resultant forces and moments of forces on sub-bodies of deformable bodies. Cauchy devised the ideal *Cauchy's cut* and corresponding contact interactions: he supposed to remove a part of a body and to replace this part with *an equivalent* system of forces, which are able to maintain the body in equilibrium. In fact, there may be a number of mathematical abstract concepts, in a model, which have the sole purpose of bringing together the various observable pieces of a theory, but having no relevance from an observational point of view. But the quantities at the basis of the model must be measurable: in the case of Mechanics, these measurable quantities are the kinematical ones. One remarks here that it is impossible to imagine an experiment able to measure the Cauchy's stress vector. Cauchy started from some *ad hoc* assumptions, like so-called *Cauchy postulate*, which is not a postulate, with the same logical status as the Principle of Virtual Work, but, on the contrary, a constitutive assumption [71,72,95,137,140–144]. Then he continued by postulating balance of forces and of moments of forces, proving the existence of a stress tensor by means of which he wrote a particular form of the Principle of Virtual Work that, for him, became a theorem. Therefore, in Truesdellian orthodoxy one finds oxymora like: *the theorem of the Principle of Virtual Work*. For continua whose deformation energy depends only on the first gradient of displacement, one could believe that d'Alembertian postulation of Cauchy postulation are two equivalent points of view. However, if one lists the higher number of basic assumptions needed to develop Cauchy postulations, when compared with those used in d'Alembertian one, then he will conclude, by using Occam razor, that the latter is much preferable. As expected, the Principle of Virtual Work postulation allows for easier generalization of the proposed models [71,72,95,140], while keeping the Cauchy postulation renders nearly impossible any generalization if not adding a long series of *ad hoc* further assumptions [145–150]. It is clear that, if one has postulated a Principle of Virtual Work and finds a series of Euler-Lagrange conditions that are logical consequence of the postulated principle, she/he will manage to repostulate the same mechanical model based on a list of balance laws, one for each independent equation obtained from the original Principle of Virtual

Work. It has to be investigated if Cauchy was aware in 1823 of the results by Piola (made public in 1822) about the nature of contact forces in first gradient continua.

At this point, some questions arise spontaneously: what is a force and how to measure it? An extremely common answer, but accepted by many with a total lack of critical spirit, is to say that *a force is what is measured using a dynamometer*. However, even the most naive scholar knows that a dynamometer measures displacements and that the value of the *measured* force provided by such an instrument is obtained by applying theoretical concepts, i.e. Hooke's law. And this, after all, simply shows that force, albeit being a very important concept, is really a merely theoretical artifice, without any direct observational meaning. No direct experimental evidence concerns forces.

In conclusion, we can say that Cauchy, Navier and Poisson decided to postulate (instead of the mathematically too difficult Principle of Virtual Work) the balance of force and moments of forces, at the cost of losing the possibility of generalizing their model: contrarily to what done by Piola, they thus only considered continua whose deformation energy depends on the first gradient of displacement. To make their postulate convincing, they then materialized the concept of force by trying to convince themselves that this mathematical concept (vector used to express a variation of energy) is intuitive and physically understandable. We believe that it is absurd that this materialization process[12] occurred in the modern age and we absolutely agree with d'Alembert (1743):

> *I have completely banned the forces associated with the body in motion, dark and metaphysical beings, capable of doing nothing more than spreading darkness over a science that is clear in itself.*

7 Conclusive remarks

In this chapter we have started a discussion about some aspects of the sociological phenomena involved in transmission and re-elaboration of scientific theories. The study of the modalities of transmission of an original theory and the involved transformation induced in the transmission process is crucial. In this context, the most problematic feature is represented by the determination of the *first* or *original* sources of scientific theories. We are not interested in a personalistic research of the *first genius* who formulated a certain theory: the formulation of scientific models is, indeed, a choral endeavor where single contributions are like small bricks in a large building. Certainly, there are lucky bricklayers who manage to contribute by building a keystone: Einstein (1879-1955) did formulate, supposedly with the help of his wife Mileva Marić (1875-1948), the basic ideas of General Relativity. However, he himself admitted that without Levi-Civita and Ricci absolute calculus he would never had the possibility to even

[12]Perhaps the similar materialization process occurred to Eudoxus' model is more understandable, considering the enormous regression of scientific knowledge that led to the Middle Ages.

write his celebrated equations. Instead, we are interested in the study of the logical process that leads to the formulation of novel and predictive theories as we believe that this study may teach us how to invent newer such theories.

A very frustrating, and often even denied, phenomenon that is systematically observed in the History of Science regards the distortion of scientific models from their original form due to the decline that periodically afflicts human societies. In fact, many historians imagine the History of Science as the accumulation of knowledge with a permanent increase of scientific understanding and technological capacity, albeit admitting that the rate of this increase has been varying in different periods. Instead, it is our belief that, unfortunately and dangerously, human technological capacity and scientific understanding of reality may experience regression.

We have argued about our thesis by presenting two clarifying examples of how, during two of these decline periods, ideas, on which the human scientific progress had been founded, were misunderstood and how even simple logical concepts could be hard to understand for scholars during the decline ages. The more common epistemological regression phenomenon which can be observed in this context consists in the complete inability to recognize the difference between a model and the object that is described by this model. This loss of clarity of thought brings, in the end, to a complete detachment of the so-called intellectuals with the world reality and technology: this detachment produces the widespread belief that theoreticians are absolutely unable to respond to demands from practical needs. Such a belief seems to ignore, just to name few Greek scientists, that Archimedes or Archytas of Tarentum did prove, with their lives, that theoretical knowledge and technology are indissolubly linked.

As a first example we have reported the case of the materialization of the Eudoxus' planetary motion model. As we have frequently remarked along the chapter, this model was not the best fruit of the Hellenistic Science, but it had strong possibilities to be pictorially represented. When the scientific knowledge of Roman society became insufficient to fully understand Aristarchus' model together with the techniques introduced by Apollonius, the Eudoxus' model, simpler from the point of view of mathematical detail, was chosen and used as a faithful representation of reality. Unfortunately, we see many modern engineers to make similar choices, with consequences that are wrongly used to discredit Scientific Engineering.

We want to stress that, in that so dark (from a knowledge point of view) historical period starting with the Roman domination on the Hellenistic colonies in Sicily, at least the study of Euclidean geometry was preserved, probably simply following a well-rooted educational tradition. As Euclidean geometry was necessary to understand and manage the few scientific elements that were left in place after the collapse of Hellenistic societies, it can be conjectured that the persistence of Euclidean geometry teaching has been probably the main reason why the early humanists were still able to interpret, or at least perceive in its importance, a very complex scientific corpus such as Archimedes'.

Eudoxus' model, which may seem naive today (albeit some flat Earth groups still believe that it is too complicated), at the time when it was introduced probably represented a conceptual revolution comparable to the formulation of the theory of caloric. Instead, the scientific advances introduced by Aristarchus' model, sticking to the same metaphor, could be considered as the advances induced by the invention of Fourier's theory of heat. With the regression of scientific awareness, due to its possibility to be pictorially represented, Eudoxus' model ended up being taken as a part of reality.

The second example we have focused on is the materialization of the abstract mathematical concept of force. In this case, as we have repeatedly remarked, the substitution process observed is more sophisticated and serious and, from certain points of view, more difficult to explain than the similar one occurred to Eudoxus' model. The materialization of force did manage to persuade many scholars that a purely mathematical object had, instead, a physical reality: this mental process seems more misleading than the materialization of rails along which planets are running. In fact, at least planet have a physical reality.

We wish to emphasize that we do not intend here to pursue a modernist attitude according to which what happened to Eudoxus's model would be less serious just because it happened about two thousand years ago. The reader will certainly be in no doubt that we are convinced that the scientific advancement of the Hellenistic age was equal, if not in some aspects superior, to that occurred in the 18th century. The less dangerous decline of Hellenistic Science is, probably, due to the fact that this decline occurred in correspondence to a socio-cultural-political reversal of enormous momentum, such as the decline of Hellenistic states and the establishment and fall of the Roman Empire. Of course, as we have repeatedly suggested, one can associate the decline of society to its scientific decline (recall the discussion about Roman aqueducts), albeit there is a time delay between the loss of scientific knowledge and the subsequent technological collapse.

The case of the materialization of the concept of force, on the other hand, is much more alarming. In fact, this confusion has occurred in a time period when scientific culture continues to develop and still manages to induce remarkable technological developments. The fact that this avant-garde science continues to develop on sometimes an extremely confused conceptual basis leads us to reflect on the possible aberrations it could produce (and it is not certain that the aberration process is not already at an advanced stage). This pessimistic view can be counterbalanced by another consideration: in present times, most likely, we have a number of active living scientists which is greater than the cumulated number of scientists who ever lived on our Earth. In fact, one can consider that the quality of this group of scientists is not as homogeneous as it was during, for instance, the flourishing of Hellenistic Science or Illuminism. Therefore, it could be that we are observing simultaneously the rise of some scientific societies in some disciplines and countries together with the decline of other scientific societies and disciplines. Therefore, the net advancement of scientific knowledge is the result of a dynamic process where declining effects are counterbalanced by development effects. In conclusion, one can say that, until the number of scientific groups that are capable to base technological advancement on solid scientific grounds is great enough, we may hope that

Dark Ages kind of decline can still be prevented.

One might say: actually, why bother with the fact that an epistemological misconception leads to the materialization of the concept of force? In the end, the concept of force is something that is used within the model anyway! No one today (apart from possibly the flat Earth groups, if they were able to understand it) could ever believe that Eudoxus' model is reality, that is, that the planets are stuck on spheres hinged to rotate rigidly relative to each other. Instead, it is commonplace, even among scientists, to believe that forces are something observable, despite so much evidence to the contrary. Let us consider a derived quantity, such as velocity: one does not need profound scientific knowledge to agree that it is not possible to measure velocity directly. Velocity can be estimated only by measuring space and time intervals, and only then one can derive an estimate of velocity from its kinematic definition. Similarly, in order to give a meaning to the concept of force, it is necessary to introduce a mathematical model: as clearly stated by d'Alembert, forces are mathematical concepts derived from basic postulates. In fact, the previously cited excerption by d'Alembert, concerning the obscurity of the concept of force, is completed by the following words[13]:

> "[...] *I must warn [the reader] that, in order to avoid circumlocutions, I often used the obscure term "force", and some other terms that are commonly employed when treating the Motion of Bodies; but I have never demanded to attach to this term other ideas than the ones resulting by the Principles that I have established, both in this Preface and in the first Part of this Treatise"* [d'Alembert, Traité de dynamique, 1743]

Forces are introduced in Mechanics as those vectors by means of which we can calculate virtual work using their inner product with virtual velocities: a very abstract definition indeed! Piola's theorem establishes a necessary condition for equilibrium, i.e the condition imposing that the resultant force and resultant moment of forces are both vanishing in an equilibrium configuration: a very abstract property indeed!

The so-called direct measures of forces are not direct at all. One measures other quantities and via the used theory the searched value of the force which she/he feels the need to talk about are determined. In a dynamometer one measures the deformation of, for example, a spring, then by further theoretical hypotheses (for example, one assumes that the measured deformation is purely elastic) and, finally, through the model (in the chosen example, Hooke's law) the *value* of the force can be calculated. If one presents the argument in this way, it is finally evident that force is a mere abstract object belonging to the used mathematical model.

Looking closely at the ontological misconception concerning the armillary

[13]The translation from French of this excerpt is by the authors of this Chapter. This is also true, in general, unless otherwise specified, for all other translations from Italian, French and Latin presented in this Chapter.

spheres and the concept of force, the question naturally arises: why does the confusion between reality and model happen so often and remain so widespread?

As it will become clear from reading the following chapters, scientific progress has often been held hostage by power groups who, for political and power related reasons (or for mere ignorance), have blocked the development of certain ideas in favor of others. The fact that Newton's equations are common knowledge in the scientific world, while Euler-Lagrange's equations are still seen as an unnecessary mathematical complication, gives us a really clear indication of how and why scientific progress in Continuum Mechanics has taken the unfortunate path that we have described.

The present chapter intentionally serves as a (long) introduction to this Volume, whose ultimate aim is to analyze the influence of modalities of sources transmission on the development of scientific theories. As it will become clearer to the reader once she/he will be engaged in the various themes we have chosen to develop in this Volume, the role of sources transmission phenomena in the development of Science is central. This fact is certainly obvious in its positive aspects: Einstein could not have written the equations of Special Relativity without Poincaré (1854-1912) or those of General Relativity without Levi-Civita's and Ricci's contributions to differential geometry. However, the sources transmission modalities play an important role also in contexts involving less *noble* scientifically and much less humanly edifying actions. This circumstance emerges very clearly, for instance, from Heiberg's text of the Prolegomena of his critical edition of the Archimedean opus, and it has strongly prompted us to offer an English translation of its more relevant passages, which otherwise remained readable only in Latin. Heiberg demonstrates with philological methods how the process of transmission of a work is by no means simple and, indeed, is conditioned by a myriad of successive modifications and alterations. An aspect that Heiberg underlines in the text of the Prolegomena, and that we are sure will strike the reader, consists in the philological deduction, that Heiberg only suggests but that Marshall Clagett [151] (and other modern scholars [30, 152]) clearly demonstrates, of the fact that in his presumed translation into Latin of Archimedes' work Niccolò Tartaglia (1499/1500-1557) heavily used an earlier translation, due to William of Moerbeke (1215/35-1286 AD), without ever mentioning him. This is very striking because it is not unique in the History of Science: periodically, someone appropriates the results of others, probably relying on the scarcity of available sources and on linguistic barriers. For example, the only copy of Moerbeke's translation, which even seems to be autograph, has been longly lost and was only found in 1885 by the German classicist Valentin Rose (1829-1916) bound to other texts. Sometimes, some sources are only available in a certain language different from the current *lingua franca* and this circumstance, unfortunately, constitutes a insurmountable barrier for a large part of the scientific community: indeed, it is as if somebody were exploiting purposely linguistic barriers for hiding the true origins of sources. We have mentioned in this chapter, and will discuss in detail in a later chapter, the sad fate of Gabrio Piola's works, forgotten for about a century and a half, just because they were written in Italian. Forgotten or willingly ignored? The reader is invited to consider that the libraries of the most important universities in the world contained copies of Piola's works, and that even theories such as Peridynamics, which Piola introduced in the 19th century, were rediscovered in the 21th century.

We believe that the studies and analyses we presented in this work can be useful for those who want to seriously approach the study of Science, not stopping at the external appearance and the universally accepted version of its development. We hope that in the future it will no longer be possible for some people to deliberately steal the work of others (sometimes even without understanding it, and therefore distorting it), hiding or destroying the name of the true authors. Today we know, although the official version still struggles to recognize them, of the invaluable contributions of Archytas, William of Moerbecke, Piola and many others. We are certain that the memory of many other scholars has been completely destroyed. It is to them that we want to dedicate our work.

Appendix: A Literary support to our theses

Albeit we tried to argue carefully about our point of view for the necessary revisitation of the History of Mechanics, we are aware that many criticisms may be attracted by the content of this chapter. In fact, in order to avoid to be considered inappropriate, many scholars preferred to insinuate some of our previous statements by using the artifice of hiding them in literary works, sometimes in the field of Science Fiction. Our attention has been particularly attracted by the masterpiece of Alfred Bester: *The stars, my destination.* We quote here some of the most relevant excerptions, in the sense we have specified, of this work.

> *BETWEEN MARS AND JUPITER is spread the broad belt of the asteroids. Of the thousands, known and unknown, most unique to the Freak Century was the Sargasso Asteroid, a tiny planet manufactured of natural rock and wreckage salvaged by its inhabitants in the course of two hundred years.*
>
> *They were savages, the only savages of the twenty-fourth century; descendants of a research team of scientists that had been lost and marooned in the asteroid belt two centuries before when their ship had failed. By the time their descendants were rediscovered they had built up a world and a culture of their own, and preferred to remain in space, salvaging and spoiling, and practicing a barbaric travesty of the scientific method they remembered from their forebears. They called themselves The Scientific People. The world promptly forgot them.*
>
> *S.S. "Nomad" looped through space, neither on a course for Jupiter nor the far stars, but drifting across the asteroid belt in the slow spiral of a dying animalcule. It passed within a mile of the Sargasso Asteroid, and it was immediately captured by The Scientific People to be incorporated into their little planet. They found Foyle.*

He awoke once while he was being carried in triumph on a litter through the natural and artificial passages within the scavenger asteroid. [...]

A crowd around the litter was howling triumphantly. "Quant Suff!" they shouted. A woman's chorus began an excited bleating: Ammonium bromide gr .11/2 Potassium bromide gr .3 Sodium bromide gr .2 Citric acid quant. suff. "Quant Suff!" The Scientific People roared. "Quant Suff!" Foyle fainted. [...]

The distant sun blazed through; the air was hot and moist. Foyle gazed around dimly. A devil face peered at him. Cheeks, chin, nose, and eyelids were hideously tattooed like an ancient Maori mask. Across the brow was tattooed JOSEPH. The "0" in JOSEPH had a tiny arrow thrust up from the right shoulder, turning it into the symbol of Mars, used by scientists to designate male sex.

"We are the Scientific Race," Joseph said. "I am Joseph; these are my people." He gestured. Foyle gazed at the grinning crowd surrounding his litter. All faces were tattooed into devil masks; all brows had names blazoned across them. [...]

"You are the first to arrive alive in fifty years. You are a puissant man. Very. Arrival of the fittest is the doctrine of Holy Darwin. Most scientific."

"Quant Suff!" the crowd bellowed.

Joseph seized Foyle's elbow in the manner of a physician taking a pulse. His devil mouth counted solemnly up to ninety-eight.

"Your pulse. Ninety-eight-point-six," Joseph said, producing a thermometer and shaking it reverently. "Most scientific."

"Quant Suff!" came the chorus. Joseph proffered an Erlenmeyer flask. It was labeled: Lung, Cat, c. s., hematoxylin & eosin. "Vitamin?" Joseph inquired. When Foyle did not respond, Joseph removed a large pill from the flask, placed it in the bowl of a pipe, and lit it. He puffed once and then gestured. Three girls appeared before Foyle. Their faces were hideously tattooed. Across each brow was a name: JOAN and MOIRA and POLLX. The "0" of each name had a tiny cross at the base.

"Choose." Joseph said. "The Scientific People practice Natural Selection. Be scientific in your choice. Be genetic."

[Alfred Bester, "The stars my destination"]

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