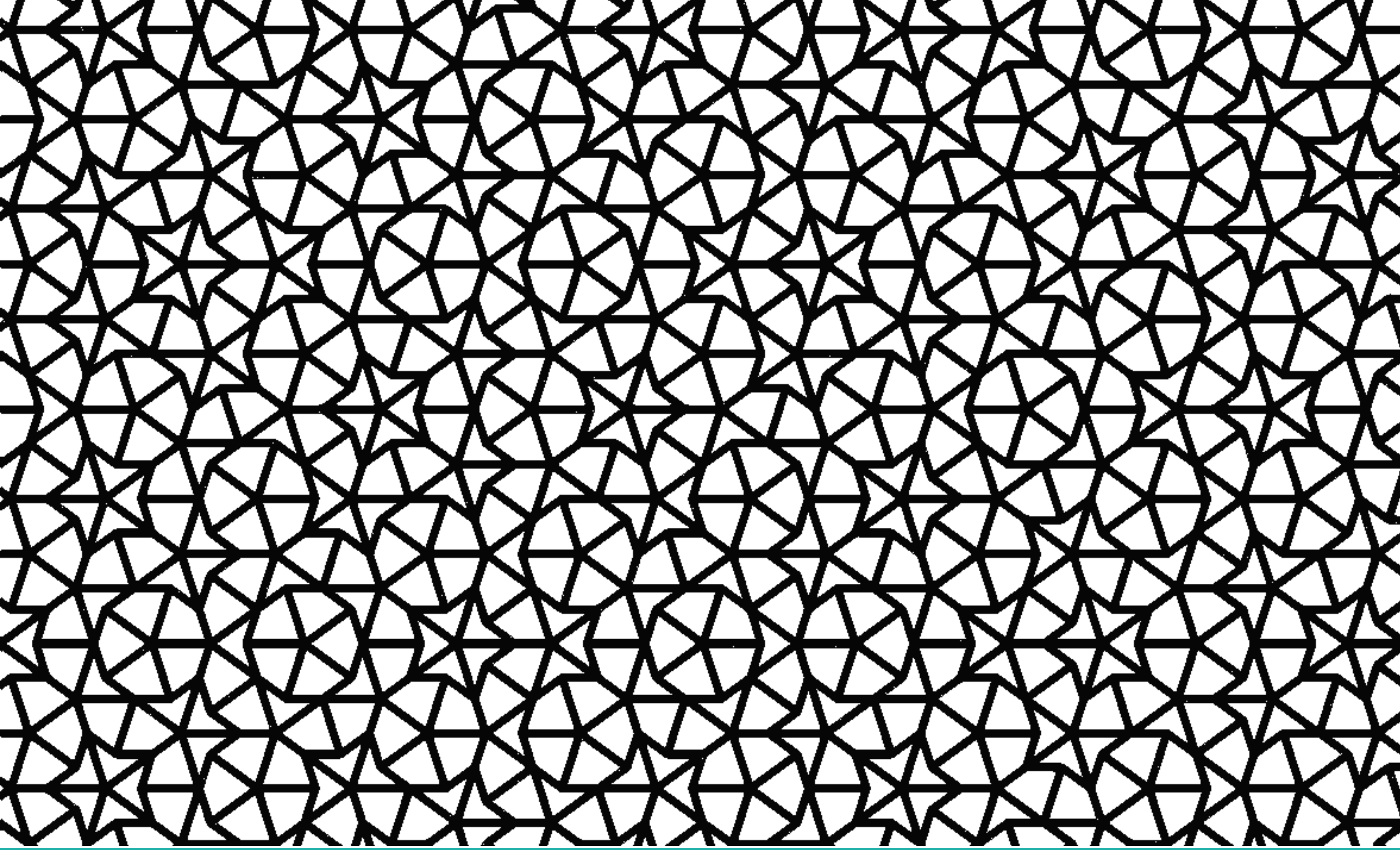

Elements de théorie des groupes pour la mécanique des solides et des structures

Document en cours de réalisation

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Contents

Introduction

This document is an elementary introduction to the theory of group representation for solid mechanics.

The aim is to provide the basic tools for understanding methods based on the exploitation of symmetries in the mechanics of materials and structures. Although these methods are very common in various fields of mathematical physics, they are still little used in our scientific fields. As they are little used, they are little taught and as they are no longer taught, they are no longer part of the methodological toolkit of the solid mechanics researcher/engineer.

The presentation is informal and we refer you to the specialist literature for further developments and for proofs of the various results we will be presenting.

One of the difficulties with this approach is that the different concepts we are going to introduce have different names depending on the community that uses them. This adds to the confusion when it comes to training in these methods. Especially as some of the classic mathematical designations can be misleading to the mechanic. For example, nothing obliges an **isotropy subgroup** to be isotropic.... The second pitfall is that examples that are obvious in one community may be perfectly cryptic in another. This is particularly true for the study of symmetry, which is often treated in the literature in relation to its applications in quantum physics.

In the presentation that follows, we have made the following choices:

- to illustrate the different notions about objects that are specific to our community;
- to give the different names of objects when these vary from one community to another.

The main bibliographic references used for the redaction of this course are:

- Kosmann-Schwarzbach, Y. (2010). Groups and symmetries. New York: Springer ;
- Baker, A. (2003). Matrix groups: An introduction to Lie group theory. Springer Science & Business Media.

Chapter 1 Fundamentals of Group Theory

Content

The aim of this first chapter is to introduce the notion of group and the elementary properties of these objects. This will also allow us to introduce the basic vocabulary and tools that will be used in the rest of this document.

We will essentially limit ourselves to notions and definitions concerning finite groups. This will avoid technical discussions of little interest to our final objective. We will, however, make a few exceptions with the use of the orthogonal group, which is a continuous group, but for which the notions introduced extend directly to the finite group.

In this chapter we will consider finite groups $|\mathrm{G}| < \infty$ or orthogonal groups. Orthogonal groups are part of continuous compact Lie groups which are a natural generalization of finite groups. Compact Lie groups will not be explicitly defined here and authors refer the curious reader to any representation theory textbook for more details.

In the following, the example of the symmetry group of an equilateral triangle has been chosen to illustrate the various definitions.

1 Elementary Definitions

1.1 Groups and subgroups

A **group** $(\mathrm{G}, *)$ is a collection (or set) of elements $\mathrm{G} = \{g_1, ..., g_n\}$ together with an operation $(*)$ with respect to which the set is closed, i.e. it maps the set to the set.

Definition 1.1 (Group)

A set $(\mathrm{G}, *)$ *is a group if:*

- ***closure**: for any* $g,\ h \in \mathrm{G}$, $g * h \in \mathrm{G}$;
- ***associativity**:*
$$(g * h) * k = g * (h * k), \quad g, h, k \ \in \mathrm{G};$$
- ***identity element**:*
$$\exists e \in \mathrm{G},\ \forall g \in \mathrm{G}, \quad \textit{such that} \quad e * g = g * e = g;$$
- ***inverse elements**:*
$$\forall g \in \mathrm{G},\ \exists h \in \mathrm{G}, \quad \textit{such that} \quad g * h = h * g = e.$$

  The element h *inverse of* g *is denoted* g^{-1}.

Note *Depending on the context, the internal law can be expressed additively* $(G, +)$ *or multiplicatively* $(G, \times)$*. When the context is clear, only the set of the group will be indicated and the notation* gh *will be used instead of* $g * h$*.*

Remark The identity element e is unique, and the inverse g^{-1} is unique for each $g \in \mathrm{G}$. It is possible for the inverse element to be the same as the element which it inverses (meaning $h^{-1}h = hh = e$). This is the case for mirror symmetry elements of a symmetry group, for instance.

Notations: We introduce here some notations which will be used throughout this course.

- $\mathbf{r}_p$ indicates a rotation in $\mathbb{R}^2$ of angle $\theta = \frac{2\pi}{p}$;

- $\mathbf{r}_p(\underline{\mathrm{n}})$ indicates a rotation in $\mathbb{R}^3$ of angle $\theta = \frac{2\pi}{p}$ along the axis $\underline{\mathrm{n}}$;
- $\boldsymbol{\pi}_n$ indicates a mirror symmetry through the line normal to $\underline{\mathrm{n}}$.

Example 1.1 Consider the symmetry group of an equilateral triangle in the plane. The isometries that leave the equilateral triangle unchanged are the identity (i.e. not changing anything), two rotations $\mathbf{r}_3$ and $\mathbf{r}_3^2$ of angles $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$, respectively and three mirror symmetries, $\boldsymbol{\pi}$, $\boldsymbol{\pi}^{(2)}$ and $\boldsymbol{\pi}^{(3)}$. They are illustrated on Figure 1.1. This group is composed of the following elements:$\{e, \mathbf{r}_3, \mathbf{r}_3^2, \boldsymbol{\pi}, \boldsymbol{\pi}^{(2)}, \boldsymbol{\pi}^{(3)}\}$.

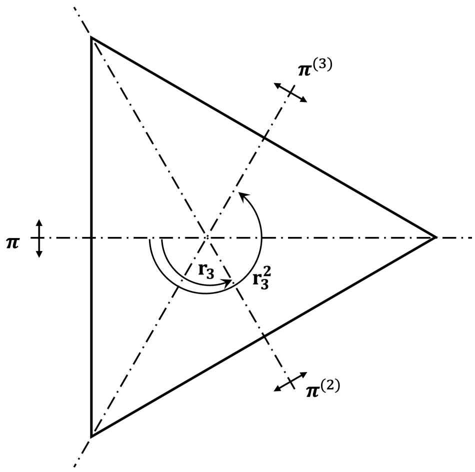


Figure 1.1: Isometries of equilateral triangle in the plane.

The elements of a group can be expressed in different manners, for instance the following set is equivalent to the former $\{e, \mathbf{r}_3, \mathbf{r}_3^2, \boldsymbol{\pi},\ \mathbf{r}_3\boldsymbol{\pi}, \mathbf{r}_3^2\boldsymbol{\pi}\}$.

Example 1.2 We can look at a more algebraic example by considering the eigenvalues of a symmetric matrix $\mathbb{R}$ of dimension 3.

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}, \qquad \text{with } \alpha < \beta < \gamma.$$

The transformations which permute these eigenvalues form a group denoted by $\mathfrak{S}_3$ which the **symmetric group** of permutations over 3 symbols. The transformation that permutes α and β is denoted (12) since it the first element and the second are exchanged, applying this logic to all permutation of (α, β, γ) we obtain

set	(α,β,γ)	(β,α,γ)	(γ,β,α)	(α,γ,β)	(β,γ,α)	(γ,α,β)
cycle	e	(12)	(13)	(23)	(123)	(213)

This group is composed of the following elements:$\{e, (12), (13), (23), (123), (213)\}$. Although it may seem different at first glance, this group is in fact the same as the one that leaves the equilateral triangle invariant. This can be seen by labelling the vertices of the previous triangle with α, β and γ. The notion of group isomorphism, introduced in the next section, will clarify this point.

Remark In these two examples, we are in fact analysing groups not as an abstract entity but as a set of transformations acting on an object, in this case a triangle or a matrix. This is a very classic approach, and its systematic study is called representation theory. We'll come back to this in a later chapter.

Definition 1.2 (Order)

The number of elements of the group G *is called the* ***order*** *of the group and is written* $|\mathrm{G}|$. *If* $|\mathrm{G}| < \infty$ *the group is said to be finite.* ♣

Definition 1.3 (Subgroup)

A subset $\mathrm{H} \subset \mathrm{G}$ *which is closed under the product of* G *is also a group. It is a subgroup* H *of* G *denoted* $\mathrm{H} < \mathrm{G}$. ♣

G always contains two subgroups, called the **trivial subgroups**, which are the group G itself and the subgroup reduced to the identity element $\{e\}$. A non-trivial subgroup H of G is said to be a **proper subgroup**.

We have the following elementary theorem relating the order of H to the order of G:

Theorem 1.1 (Lagrange)

If H *is a subgroup of* G. $|\mathrm{H}|$ *divides* $|\mathrm{G}|$, *so the index of* $|\mathrm{H}|$ *in* $|\mathrm{G}|$ *is an integer.* ♡

This index of H in G is usually denoted by $[\mathrm{G} : \mathrm{H}]$ or $\frac{|\mathrm{H}|}{|\mathrm{G}|}$.

1.2 Homomorphism

The transformations which preserve an object depend strongly in their expression on the nature of the object considered: geometric figure, vector, function,. . . However, the rules that bind the composition of these transformations can be identical despite very different contexts. The notion of an abstract group focuses on the identity of relations within a set, without any contextual interpretation.

To study the relationships that exist between different groups, we need to introduce the notion of homomorphism, i.e. applications between two groups that preserve their group structure. This gives rise to the notion of isomorphism, which makes it possible to identify when two groups defined in different contexts correspond to the same abstract group.

Definition 1.4 (Homomorphism)

A ***homomorphism*** α *is a mapping that respects the group structure between two groups* G *and* H *such that* $\alpha : \mathrm{G} \longrightarrow \mathrm{H}$*:*

$$\alpha(g_1 g_2) = \alpha(g_1)\alpha(g_2), \; \forall g_1, g_2 \in \mathrm{G}$$

♣

Note *In addition, a homomorphism always maps the identity element* $e_{\mathrm{G}} \in \mathrm{G}$ *to the identity element* $e_{\mathrm{H}} \in \mathrm{H}$*, as*

$$\alpha(e_{\mathrm{G}})\alpha(g) = \alpha(e_{\mathrm{G}} g) = \alpha(g) = e_{\mathrm{H}}\alpha(g), \; \forall g \in \mathrm{G}$$

A bijective homomorphism between two groups can sometimes be defined. In this case, the mapping is called an **isomorphism**. If such isomorphism between G and H exists, the two groups are called **isomorphic**.

Example 1.3 In the previous example, we said that the symmetry group of the equilateral triangle and the group of permutation of the eigenvalues of a matrix of dimension 3 were identical. This can be formalised by constructing the isomorphism $\alpha : \mathrm{D}_3 \to \mathfrak{S}_3$, where $\mathfrak{S}_3$ is the symmetric group of permutations over 3 symbols, as follows

g	e	$\boldsymbol{\pi}$	$\boldsymbol{\pi}^{(2)}$	$\boldsymbol{\pi}^{(3)}$	$\mathbf{r}_3$	$\mathbf{r}_3^2$
$\alpha(g)$	e	(12)	(13)	(23)	(123)	(213)

Definition 1.5 (Automorphism)

If an isomorphism maps G *to itself then it is called an* ***automorphism*** *of* G. *We call* $\mathrm{Aut}(\mathrm{G})$ *the set of all automorphisms of* G. ♣

In particular, the automorphism defined by $\alpha_g(h) = ghg^{-1}$, is called an **inner automorphism** or a conjugation. There exists as many inner automorphisms for a group as elements in the group.

1.3 Basic properties

An important property, to begin with, concerns the commutativity of the law of internal composition.

Definition 1.6 (Abelian group)

A group $(G, \star)$ is said to be Abelian, or commutative, when the internal composition law of the group is commutative, i.e.

$$\forall a, b \in \mathrm{G},\ a \star b = b \star a.$$

♣

The elementary properties of a finite group can be read from its **Cayley table**. This is a table in which all the possible products of all the elements of the group are given. This is also known as a **multiplication table**, with the subtlety that the group law may not be commutative, unlike classical multiplication. It results that, if the Cayley table is symmetric, the group is Abelian.

Remark

- In a Cayley table, each row/column contains 1 and only 1 copy of each element of G.
- Many properties of a group – such as whether or not it is Abelian, which elements are inverses of which elements – can be read from its Cayley table.
- A Cayley table is uniquely associated with an abstract group. As such, by looking at a Cayley table, a group can be identified.

Example 1.4 Consider $\mathrm{G} = \{g_1, g_2, g_3\}$, the Cayley table of this group is presented as follows:

	g_1	g_2	g_3
g_1	g_1	g_2	g_3
g_2	g_2	g_3	g_1
g_3	g_3	g_1	g_2

Elements g_k of the table are obtained as $g_i g_j = g_k$, with g_i the element in the line i, and g_j the element of the jth-column. On this particular example:

- g_1 is the identity element e;
- the table is symmetric hence the group is Abelian;
- any element can be generated from g_2, which verifies $g_2^3 = e$.

Hence this group is isomorphic to the abstract group $\mathbb{Z}_3$.

✍ **Exercise 1.1** Construct Cayley tables for D_3 and $\mathfrak{S}_3$. Check that these two groups are isomorphic.

1.4 Generating sets

It can be observed that, using the internal law, the complete set of elements of a group G can be generated from a restricted set of elements $\mathrm{P} \subset \mathrm{G}$ called the **set of generators.** This means that each element $g \in \mathrm{P}$ can be written as a product of a finite number of elements[1] of P. A set of generators is said to be minimal if none of its subsets generates G. We denote by $< g, h, \ldots >$ the group generated by g, h, ...

Note *Generating sets, even minimal ones, are not uniquely defined and so cannot be used to compare different groups.*

[1]This must be understood modulo the multiplication by the identity to any power.

Remark It should be noted that it is customary in some physical communities to define groups in terms of a non-minimal set of generators mostly for practical computational reasons. In what follows, and unless explicitly stated, the sets of generators will always be minimal.

Elements multiplied by themselves can be expressed as **element powers** as follows:

$$g^2 = gg,\ g^3 = ggg, \ldots, g^n = gg^{n-1}$$

The order of an element is the smallest n such as $g^n = e$. The group of order n generated by $< g >$ is said to be cyclic.

Definition 1.7 (Cyclic group)

The abstract cyclic group, denoted $\mathbb{Z}_n$, is the monogeneous group of order n. ♣

It can be directly checked that cyclic groups are Abelian. The converse is false, but cyclic groups are essential ingredients for generating any commutative group[2].

Example 1.5 In the context of linear isometry of $\mathbb{R}^2$, $\mathbb{Z}_n$ is used to model the rotational invariance of a figure since $< \mathbf{r}_n >$ is isomorphic to $\mathbb{Z}_n$.

Example 1.6 A less standard example is the following. Let $S^2(\mathbb{R}^d)$ denote the space of symmetric second order tensors and $\mathcal{L}(S^2(\mathbb{R}^d))$ denote the space of linear applications on $S^2(\mathbb{R}^d)$. An element $\underset{\approx}{\mathrm{T}} \in \mathcal{L}(S^2(\mathbb{R}^d))$ is a fourth-order tensor which, when expressed in components with respect to a basis, exhibits the following index symmetries:

$$T_{ijkl} = T_{jikl} = T_{jilk} = T_{ijlk}.$$

These so-called minor symmetries can be indicated as $T_{(ij)(kl)}$, where the notation in parenthesis (ij) is used to indicate a permutation symmetry of indices i and j. The set of index permutations letting a tensor invariant is called the **index symmetry group**, in the present case this is an Abelian group isomorphic to[a] $\mathbb{Z}_2 \times \mathbb{Z}_2$.

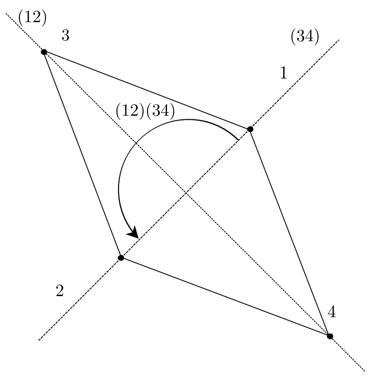


Figure 1.2: The index symmetry group $\mathbb{Z}_2 \times \mathbb{Z}_2$ generated by the transposition (12) and (34)

[a] This group is also known as Klein's group or Vierergruppe. We can see that this group is isomorphic to D_2, but that it is commutative, unlike the higher-order dihedral groups.

There is another very important elementary abstract group, which is known as the dihedral group.

Definition 1.8 (Dihedral group)

*The abstract dihedral group, denoted $\mathbb{D}_n$ is a group of order $2n$ generated from two elements p, q verifying the following **presentation***

$$p^n = q^2 = e, \quad pq = qp^{-1}$$

♣

[2] The fundamental theorem of finite Abelian groups states that every finite Abelian group G can be expressed as the direct sum of cyclic subgroups of prime-power order.

The last relation in the presentation of the group indicates that dihedral group are not commutative for $n > 2$.

Note *It is important to distinguish between the abstract structure of a group and its* ***interpretation in a particular context****. For example, in the context of linear isometries of $\mathbb{R}^2$, we can consider*

- *the π-angle rotation, denoted $\mathbf{r}_2$, an operation with determinant +1;*
- *the mirror transformation with respect to $\underline{\mathrm{n}}$, denoted $\boldsymbol{\pi}(\underline{\mathrm{n}})$, with determinant -1.*

They generate, respectively the following groups of order two: $<\mathbf{r}_2>=\{\mathrm{e},\mathbf{r}_2\}$ and $<\boldsymbol{\pi}(\underline{\mathrm{n}})>=\{\mathrm{e},\boldsymbol{\pi}(\underline{\mathrm{n}})\}$. In a particular physical context, these groups can be given different names to insist on their content. Here the following notation are customary

$$<\mathbf{r}_2>=\mathrm{Z}_2, \quad <\boldsymbol{\pi}(\underline{\mathrm{n}})>=\mathrm{D}_1$$

but as abstract groups they are identical and isomorphic to $\mathbb{Z}_2$. So, to insist, the abstract notations of groups insist on the algebraic structure of the generators and not on the interpretation they have in a given context.

Example 1.1 (Continued) Back to the continued example of the equilateral triangle, its symmetry group is the dihedral group D_3 of order 6 and generated as

$$\mathrm{D}_3=<\mathbf{r}_3,\boldsymbol{\pi}>$$

The generators satisfy the presentation relations:

$$\mathbf{r}_3^3=\mathrm{e}, \quad \boldsymbol{\pi}^2=\mathrm{e}, \quad \mathbf{r}_3\boldsymbol{\pi}=\boldsymbol{\pi}\mathbf{r}_3^{-1}$$

The group D_3 possesses 6 subgroups that are presented below:

- The identity group is the only subgroup of D_3 with order 1:
$$\mathrm{Z}_1=\{\mathrm{e}\}$$
- There are three order 2 subgroups. They are generated by one of the mirror symmetries:
$$\mathrm{D}_1^{(1)}=<\boldsymbol{\pi}>=\{\mathrm{e},\boldsymbol{\pi}\},\ \mathrm{D}_1^{(2)}=<\boldsymbol{\pi}^{(2)}>=\{\mathrm{e},\boldsymbol{\pi}^{(2)}\},\ \mathrm{D}_1^{(3)}=<\boldsymbol{\pi}^{(3)}>=\{\mathrm{e},\boldsymbol{\pi}^{(3)}\}$$
All these groups are isomorphic do the abstract group $\mathbb{Z}_2$ but are named differently due to their different physical content. The parenthesis exponents are used to differentiate these groups with respect to the mirror axis used to define their reflection operation.
- The only order 3 subgroup of D_3 is the Abelian group generated by the rotation of angle $\frac{2\pi}{3}$:
$$\mathrm{Z}_3=<\mathbf{r}_3>=\{\mathrm{e},\mathbf{r}_3,\mathbf{r}_3^2\}$$

The structure of the group D_3 is detailed in the Cayley table presented Table 1.1.
One can observe that the Cayley table is not symmetric and, as such, the group D_3 is not Abelian (or cyclic). It can also be observed that the inverse of element $\mathbf{r}_3$, $\mathbf{r}_3^{-1}$, is in fact the element $\mathbf{r}_3^2$.

	e	$\mathbf{r}_3$	$\mathbf{r}_3^2$	$\boldsymbol{\pi}$	$\boldsymbol{\pi}^{(2)}$	$\boldsymbol{\pi}^{(3)}$
e	e	$\mathbf{r}_3$	$\mathbf{r}_3^2$	$\boldsymbol{\pi}$	$\boldsymbol{\pi}^{(2)}$	$\boldsymbol{\pi}^{(3)}$
$\mathbf{r}_3$	$\mathbf{r}_3$	$\mathbf{r}_3^2$	e	$\boldsymbol{\pi}^{(3)}$	$\boldsymbol{\pi}^{(1)}$	$\boldsymbol{\pi}^{(2)}$
$\mathbf{r}_3^2$	$\mathbf{r}_3^2$	e	$\mathbf{r}_3$	$\boldsymbol{\pi}^{(2)}$	$\boldsymbol{\pi}^{(3)}$	$\boldsymbol{\pi}^{(1)}$
$\boldsymbol{\pi}$	$\boldsymbol{\pi}$	$\boldsymbol{\pi}^{(2)}$	$\boldsymbol{\pi}^{(3)}$	e	$\mathbf{r}_3$	$\mathbf{r}_3^2$
$\boldsymbol{\pi}^{(2)}$	$\boldsymbol{\pi}^{(2)}$	$\boldsymbol{\pi}^{(3)}$	$\boldsymbol{\pi}^{(1)}$	$\mathbf{r}_3^2$	e	$\mathbf{r}_3$
$\boldsymbol{\pi}^{(3)}$	$\boldsymbol{\pi}^{(3)}$	$\boldsymbol{\pi}^{(1)}$	$\boldsymbol{\pi}^{(2)}$	$\mathbf{r}_3$	$\mathbf{r}_3^2$	e

Table 1.1: Cayley table of group D_3

It can be observed on this table that if the product of two rotations is a rotation, the product of two mirrors is not a mirror.

Example 1.4 (Continued) Let's return to the example of tensor index symmetries. Let $S^2(S^2(\mathbb{R}^d))$ denote the space of symmetric linear applications on $S^2(\mathbb{R}^d)$. An element $\underset{\approx}{\mathrm{T}} \in S^2(S^2(\mathbb{R}^d))$ is a fourth-order tensor which, when expressed in components with respect to a basis, exhibits the following index symmetries:

$$T_{ijkl} = T_{jikl} = T_{jilk} = T_{ijlk} = T_{klij} = T_{klji} = T_{lkij} = T_{lkji}$$

These so-called minor and major symmetries can be summarised as $T_{\underline{\underline{(ij)}\ \underline{(kl)}}}$, where the underlined notation stands for permutation symmetry with respect to the underlined set of indices, meaning that permutation of set of indices ij with set of indices kl leaves the tensor unchanged. This major symmetries is encoded by the double transposition $(13)(24)$. The set of index permutations letting a tensor invariant is called the **index symmetry group**, in the present case this is a group isomorphic to $\mathbb{D}_4$, which is of order 8. The following generators can be considered

$$p = (1324), \quad q = (12)$$

The generator p is the 4-cycle permutation such as $1 \to 3 \to 2 \to 4$. Its successive actions on the word ijk is given by the following sequence:

$$ijkl \xrightarrow{p} klij \xrightarrow{p} jilk \xrightarrow{p} lkij \xrightarrow{p} ijkl$$

this operation verifies $p^4 = e$. The generator q is the transposition $1 \to 2$:

$$ijkl \xrightarrow{q} jikl \xrightarrow{q} ijkl$$

this operation verifies $q^2 = e$. The last relation $(pq)^2 = e$ can easily be checked. The transformations can be illustrated on the following figure.

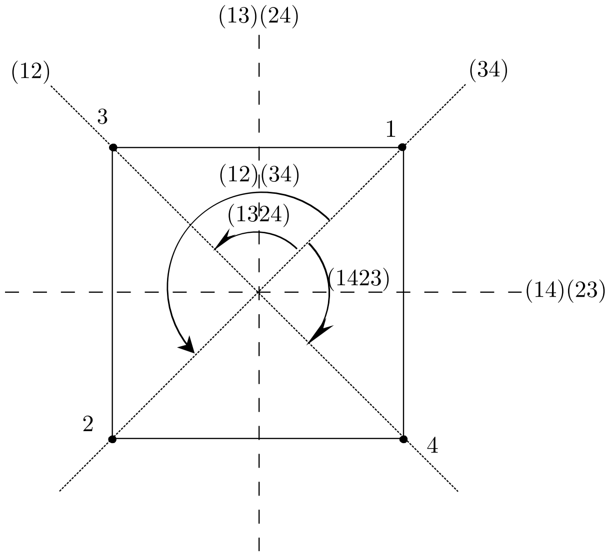


Figure 1.3: The index symmetry group D_4 of the elasticity tensor

Note that in elasticity the following generators are used instead

$$L = (13)(24), \quad q = (12)$$

but these make the $\mathbb{D}_4$ structure less obvious for the example.

1.5 Structure of groups

The structure of a group can be very rich and does not reduce to the sole collection of its subgroups and elements. One immediate observation is that not all the elements are equivalent, since their order generally differs. The same observation can be made for the set of subgroups.

To make this observation rigorous, we are going to introduce the notion of an**equivalence class**. This notion allows us to partition the elements of a group into disjoint sets with respect to a chosen notion of equivalence. Two equivalence relationships will be examined below

1. cosets classes ;
2. conjugacy classes.

Let begin by introducing the definition of an equivalence relation

Definition 1.9 (Equivalence relation)

*A binary relation $\sim$ on a set X is said to be an **equivalence relation**, if and only if it is **reflexive**, **symmetric** and **transitive**. That is, for all a, b, and c in X*
(reflexivity) : $a \sim a$;
(symmetry) : $a \sim b \Leftrightarrow b \sim a$;
(transitivity) : $a \sim b, b \sim c \Rightarrow a \sim c$

♣

Given a set X and an equivalence relation $\sim$ on X, the equivalence class of an element $a \in X$ denoted by

$$[a] = \{x \in X : x \sim a\}$$

$[a]$ is the set of elements which are equivalent to a. Equivalence classes form a partition of X, meaning that any element $x \in X$ belong to a unique class[3].

1.5.1 Coset decomposition

A subgroup H of a group G may be used to decompose the underlying set of G into disjoint, equal-size subsets called **cosets**. There are **left cosets** and **right cosets**. Cosets (both left and right) have the same number of elements as does H. Furthermore, H itself is both a left coset and a right coset. The number of left cosets of H in G is equal to the number of right cosets of H in G.

Definition 1.10 (Left Coset)

Consider H a subgroup of G. The elements of H are denoted $\{h_1, h_2, \ldots, h_m\}$, with $m = |\mathrm{H}|$, while $r_1, r_2, \ldots$ be the remaining elements of $\mathrm{G} \notin \mathrm{H}$. The set

$$r_k\mathrm{H} = \{r_k h_1, r_k h_2, \ldots, r_k h_m\}$$

is the left coset of H with respect to the element r_k.

♣

Note *The notion of right coset is defined as $\mathrm{H}r_k$ in a similar way.*

Two elements being in the same left coset defines a natural equivalence relation.

Definition 1.11

Two elements x and y of G, are said to be equivalent with respect to the subgroup H if $x\mathrm{H} = y\mathrm{H}$. The equivalence classes of this relation are the left cosets of H.

♣

The equivalence classes for this relation are defined as

$$[a] = \{x \in \mathrm{G} | x \in a\mathrm{H}\}$$

In the former definition a is said to be the representative of the coset.

Note *It should be noted that in general the cosets are not group.*

Property *Some properties of cosets:*

[3] The set of classes constitutes an disjointed covering of X.

- *every element of* G *appears either in* H *or in one of its* $r_k\mathrm{H}$*;*
- *no element can be common to two different cosets of the same subgroup;*
- *no coset can contain the same element more than once.*

All of these properties results from the fact that the equivalence classes generated by the cosets form a partition of G. As a consequence we have the following left coset decomposition of G:

$$\mathrm{G} = \mathrm{H} \cup r_1\mathrm{H} \cup \ldots \cup r_p\mathrm{H}$$

Example 1.7 Let G be a finite group acting on a figure with symmetry group H, with $\mathrm{H} < \mathrm{G}$. Some elements of G will transform the figure, others will leave it invariant. It is often important to know the action produced by elements of G on the figure. The left cosets indicate how the transformations of G are viewed by a H-invariant object.
For instance, consider the action of $\mathrm{G} = \mathrm{D}_6$ on an equilateral triangle whose symmetry group is $\mathrm{H} = \mathrm{D}_3$. Elements of D_3 are

$$\mathrm{D}_3 = \{e, \mathbf{r}_3, \mathbf{r}_3^2, \boldsymbol{\pi}, \mathbf{r}_3\boldsymbol{\pi}, \mathbf{r}_3^2\boldsymbol{\pi}\}$$

And the decomposition of D_6 into left cosets gives $\mathrm{D}_6 = \mathrm{D}_3 \cup \mathbf{r}_6\mathrm{D}_3$. In fact, the operations in D_6 are classified according to their action on the figure

$$[e] = \{e, \mathbf{r}_6^2, \mathbf{r}_6^4, \boldsymbol{\pi}, \mathbf{r}_6^2\boldsymbol{\pi}, \mathbf{r}_6^4\boldsymbol{\pi}\}$$
$$[\mathbf{r}_6] = \{\mathbf{r}_6, \mathbf{r}_6^3, \mathbf{r}_6^5, \mathbf{r}_6\boldsymbol{\pi}, \mathbf{r}_6^3\boldsymbol{\pi}, \mathbf{r}_6^5\boldsymbol{\pi}\}.$$

Example 1.8 The group acting on the indices of a 4-th order tensor is the symmetric group $\mathfrak{S}_4$, which is a group of order 24. The index symmetry group of the elasticity tensor is the group $\mathrm{D}_4^{\mathrm{ela}}$ generated by

$$p = (1324), \ q = (12).$$

We have $[\mathfrak{S}_4 : \mathrm{D}_4^{\mathrm{ela}}] = 3$ and we can carry out the left coset decomposition

$$\mathfrak{S}_4 = \mathrm{D}_4^{\mathrm{ela}} \cup (23)\mathrm{D}_4^{\mathrm{ela}} \cup (243)\mathrm{D}_4^{\mathrm{ela}}$$

In particular, we can deduce that the following permutations of the elasticity tensor are inequivalent:

$$\{C_{ijkl}, C_{ikjl}, C_{iljk}\}$$

In fact, if we want to symmetrise $\underset{\approx}{\mathrm{C}}$ we will do so on these permutations.

Remark The previous examples shows that when acting on a mathematical object on the left, the left cosets decomposition is more natural than the right one to factorise the action.

Note *Most of Computer Assisted Softwares (CAS) possess functions to compute coset decomposition. The most powerful software for group manipulation is GAP, but most of the computation can be done with Mathematica or Mapple.*

1.5.2 Conjugacy classes

Another way of partitioning the elements of a group is to join elements of the same nature together. This idea is formalize by the notion of conjugacy.

Definition 1.12 (Conjugacy)

Two elements $g, h \in \mathrm{G}$ *are conjugate,* $g \approx h$*, if there exists a group element that satisfies*

$$\exists d \in \mathrm{G}, \ g = dhd^{-1}$$

♣

When said with the hands, transformations that differ only in their orientation within the group are lumped together. For instance, all mirrors in the symmetry group D_3 belong to the same conjugation class, because they differ only in the orientation of their characteristic plane. Beware however that if one considers the symmetry group of a regular hexagon, the mirrors with axis passing through the vertex are in a different conjugation class as the ones with axis massing through the middle of one side. Similarly, rotations of a given order constitute a conjugacy class, since rotations differ only in the orientation of their characteristic axis.

Example 1.1 (Continued) Back to our continued example on the equilateral triangle.
The various (left) cosets of symmetry group D_3 are the following:

- with respect to $\mathrm{D}_1^{(1)}$,
$$\mathrm{D}_3 = \mathrm{D}_1^{(1)} \cup \mathbf{r}_3\mathrm{D}_1^{(1)} \cup \mathbf{r}_3^2\mathrm{D}_1^{(1)}$$
with
$$\mathrm{D}_1^{(1)} = \{\mathrm{e}, \boldsymbol{\pi}^{(1)}\}, \quad \mathbf{r}_3\mathrm{D}_1^{(1)} = \{\mathbf{r}_3, \boldsymbol{\pi}^{(3)}\}, \quad \mathbf{r}_3^2\mathrm{D}_1^{(1)} = \{\mathbf{r}_3^2, \boldsymbol{\pi}^{(2)}\}$$
- with respect to $\mathrm{D}_1^{(2)}$,
$$\mathrm{D}_3 = \mathrm{D}_1^{(2)} \cup \mathbf{r}_3\mathrm{D}_1^{(2)} \cup \mathbf{r}_3^2\mathrm{D}_1^{(2)}$$
with
$$\mathrm{D}_1^{(2)} = \{\mathrm{e}, \boldsymbol{\pi}^{(2)}\}, \quad \mathbf{r}_3\mathrm{D}_1^{(2)} = \{\mathbf{r}_3, \boldsymbol{\pi}^{(1)}\}, \quad \mathbf{r}_3^2\mathrm{D}_1^{(2)} = \{\mathbf{r}_3^2, \boldsymbol{\pi}^{(3)}\}$$
- with respect to $\mathrm{D}_1^{(3)}$,
$$\mathrm{D}_3 = \mathrm{D}_1^{(3)} \cup \mathbf{r}_3\mathrm{D}_1^{(3)} \cup \mathbf{r}_3^2\mathrm{D}_1^{(3)}$$
with
$$\mathrm{D}_1^{(3)} = \{\mathrm{e}, \boldsymbol{\pi}^{(3)}\}, \quad \mathbf{r}_3\mathrm{D}_1^{(3)} = \{\mathbf{r}_3, \boldsymbol{\pi}^{(2)}\}, \quad \mathbf{r}_3^2\mathrm{D}_1^{(3)} = \{\mathbf{r}_3^2, \boldsymbol{\pi}^{(1)}\}$$
- with respect to Z_3,
$$\mathrm{D}_3 = \mathrm{Z}_3 \cup \boldsymbol{\pi}^{(1)}\mathrm{Z}_3$$
with
$$\mathrm{Z}_3 = \{\mathrm{e}, \mathbf{r}_3, \mathbf{r}_3^2\}, \quad \boldsymbol{\pi}^{(1)}\mathrm{Z}_3 = \{\boldsymbol{\pi}^{(1)}, \boldsymbol{\pi}^{(2)}, \boldsymbol{\pi}^{(3)}\}$$

One can check the properties of cosets. For instance if one considers the cosets of D_3 with respect to $\mathrm{D}_1^{(1)}$, it can be observed that all six elements of D_3 are either contained in $\mathrm{D}_1^{(1)}$ or in one of its cosets and that no element is common to two different cosets.
There are 3 conjugacy classes for the dihedral group D_3:

- identity: $\{\mathrm{e}\}$;
- 2-cycle: $\{\boldsymbol{\pi}^{(1)}, \boldsymbol{\pi}^{(2)}, \boldsymbol{\pi}^{(3)}\}$;
- 3-cycle: $\{\mathbf{r}_3, \mathbf{r}_3^2\}$.

The notion of conjugacy can naturally be extended from elements to subgroups.

Definition 1.13 (Conjugate subgroup)

Let H, K *be two subgroups of* G, H *and* K *are conjugate with respect to* G, *denoted* $\mathrm{H} \sim \mathrm{K}$ *if*
$$\mathrm{K} = \{ghg^{-1} |\ h \in \mathrm{H}\} = g\mathrm{H}g^{-1} \quad \textit{for some } g \in \mathrm{G}$$
♣

We have the following lemma which simplify the computation of conjugate groups

Lemma 1.1

Let H, K *be two conjugate subgroups of* G,
$$\mathrm{K} = g\mathrm{H}g^{-1} \quad \textit{for some } g \in \mathrm{G}$$

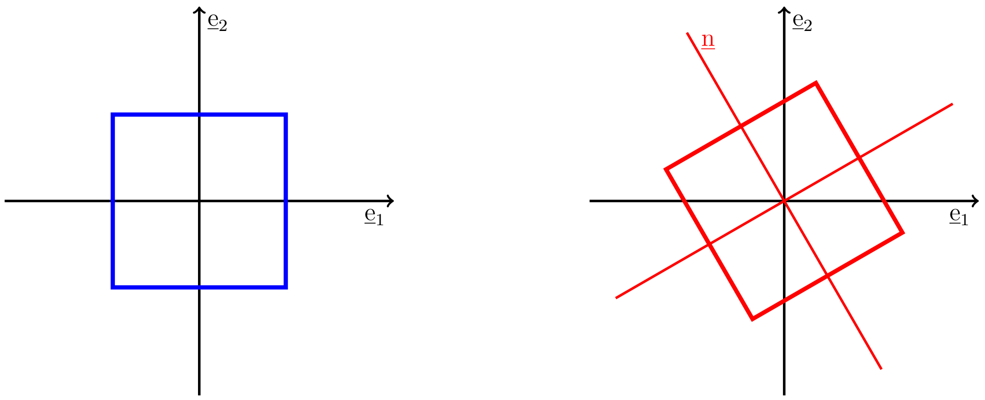
Conjugate groups are generated by conjugate set of generators, i.e.

$$\mathrm{K} =< gp_1 g^{-1}, \ldots, gp_n g^{-1} >$$

with $(p_1, \ldots, p_n)$ *a set of generators for* H.

Roughly speaking, conjugate groups are identical up to the orientation of their generators.

Note *Even though conjugate groups are isomorphic, the converse implication is false. Consider, for instance, in the context of plane symmetry, the group generated by a mirror transformation, denoted* D_1 *and the group* Z_2 *generated by a* π *angle rotation. These groups are isomorphic, but not conjugate.*



(a) $\mathrm{D}_4^{\underline{e}_2}$ **(b)** $\mathrm{D}_4^{\underline{n}}$

Figure 1.4: An example of figures having conjugate symmetry group.

Definition 1.14 (Subgroup-class)

The subgroup-class of a subgroup is denoted $[\mathrm{H}]$ *where* $\mathrm{H} \subseteq \mathrm{G}$ *is a class representative. That is*

$$[\mathrm{H}] = \{g\mathrm{H}g^{-1} | g \in \mathrm{G}\}$$

the set of all subgroups of G *that are conjugate to* H.

Subgroup-class induces a partition among subgroups of G, meaning that any subgroup of G belongs to exactly one these classes.

Note *In the context of physics,* ***subgroup classes*** *are often referred to as* ***symmetry classes****. This is an important concept, because it characterises the symmetry of an object independently of its orientation in space. In material science when someone, for instance, said the elastic behaviour of the wood is orthotropic, this refers to the symmetry class of the constitutive law, not its symmetry group.*

Example 1.1 (Continued) Back to the dihedral group D_3, its subgroups classes are the following:

- $[1] = \{1\}$
- $[\mathrm{D}_1] = \{\mathrm{D}_1^{(1)}, \mathrm{D}_1^{(2)}, \mathrm{D}_1^{(3)}\}$
- $[\mathrm{Z}_3] = \{\mathrm{Z}_3\}$

One can check that, for instance, subgroup $\mathrm{D}_1^{(2)}$ generated by $< \boldsymbol{\pi}^{(2)} >$, has a generator which is conjugated to the one of $\mathrm{D}_1(1)$. Indeed, $\boldsymbol{\pi}^{(2)} = \mathbf{r}_3^2 \boldsymbol{\pi} \mathbf{r}_3$, with $\mathbf{r}_3 = (\mathbf{r}_3^2)^{-1}$.

1.5.3 Factor subgroup

In this final section, we consider a situation in which the two previous decompositions meet. To that aim let us introduce a very special kind of subgroup:

Definition 1.15 (Invariant or Normal subgroups)

A subgroup H *of* G *is said to be a* G*-invariant, or* ***normal****, subgroup, denoted* $\mathrm{H} \triangleleft \mathrm{G}$*, if*

$$\forall g \in \mathrm{G},\ g\mathrm{H}g^{-1} = \mathrm{H}$$

♣

Equivalently we have the property that, for $\mathrm{H} \triangleleft \mathrm{G}$,

$$\forall g \in \mathrm{G},\ g\mathrm{H} = \mathrm{H}g$$

meaning that the left and the right cosets are identical. We have the following helpful lemma

Lemma 1.2

A subgroup of index 2 is always normal.

♡

Proof Suppose H is a subgroup of G of index 2. Then there are only two cosets of G relative to H. Let $s \in Gr\ \mathrm{H}$. Then G can be decomposed into the cosets H, $s\mathrm{H}$ or H, $\mathrm{H}s$, implying H commutes with s. Since $\mathrm{H}h = h\mathrm{H}$ for any $h \in \mathrm{H}$, we see that H commutes with every element of G and hence is normal. □

Definition 1.16 (Factor Group)

Let H *be normal in* G, $\mathrm{H} \triangleleft \mathrm{G}$*, the set of all cosets is a group called the* ***Factor Group*** *and denoted* G/H.

♣

Proof Let's proove the factor set is in fact a group. Consider the left cosets of H,

$$\mathrm{G}/\mathrm{H} = \{\mathrm{H}, r_2\mathrm{H}, \ldots, r_m\mathrm{H}\}$$

Consider the product of two elements

$$(r_i\mathrm{H})(r_j\mathrm{H}) = r_ir_j\mathrm{H} = r_k\mathrm{H}$$

□

Example 1.1 (Continued) It can be observed that $\mathrm{Z}_3 \triangleleft \mathrm{D}_3$. Since Z_3 is a subgroup of D_3 of order 2, it is a normal subgroup. It results that $\mathrm{D}_3/\mathrm{Z}_3$ is a group called the factor group.

Factor group: $\mathrm{D}_3/\mathrm{Z}_3$ By definition

$$\mathrm{D}_3/\mathrm{Z}_3 = \{\mathrm{eZ}_3, \boldsymbol{\pi}^{(1)}\mathrm{Z}_3\}$$

Hence

$$(\boldsymbol{\pi}^{(1)}\mathrm{Z}_3)(\boldsymbol{\pi}^{(1)}\mathrm{Z}_3) = \mathrm{eZ}_3$$

and more generally

$\mathrm{D}_3/\mathrm{Z}_3$	Z_3	$\boldsymbol{\pi}^{(1)}\mathrm{Z}_3$
Z_3	Z_3	$\boldsymbol{\pi}^{(1)}\mathrm{Z}_3$
$\boldsymbol{\pi}^{(1)}\mathrm{Z}_3$	$\boldsymbol{\pi}^{(1)}\mathrm{Z}_3$	Z_3

Isomorphism Consider the group $\mathbb{Z}_2$ defined by the following Cayley table

$\mathbb{Z}_2$	e	$\mathbf{r}_2$
e	e	$\mathbf{r}_2$
$\mathbf{r}_2$	$\mathbf{r}_2$	e

And let the group morphism $\phi : \mathrm{D}_3/\mathrm{Z}_3 \Rightarrow \mathbb{Z}_2$ defined by the relation

$$\phi(\mathrm{Z}_3) = \mathrm{e}, \quad \phi(\boldsymbol{\pi}^{(1)}\mathrm{Z}_3) = \mathbf{r}_2$$

This provides the following isomorphism

$$\mathrm{D}_3/\mathrm{Z}_3 \simeq \mathbb{Z}_2$$

1.6 Composition of groups

In this section, we'll look at different processes that can be used to create larger groups from smaller ones. The approach can also be seen in the opposite direction, showing how to break up a large group as a collection of more elementary groups.

Definition 1.17 (Direct Product)

The direct product of two groups $(\mathrm{G}, \star)$ *and* $(\mathrm{H}, +)$ *is a group denoted,* $(\mathrm{G} \times \mathrm{H}, \circ)$*, defined as*

$$\mathrm{G} \times \mathrm{H} = \{(g, h) \mid g \in \mathrm{G} \text{ and } h \in \mathrm{H}\}$$

with the group operation

$$(g_1, h_1) \circ (g_2, h_2) = (g_1 \star g_2, h_1 + h_2), \quad \forall g_1, g_2 \in \mathrm{G}, \forall h_1, h_2 \in \mathrm{H}$$

Since G*,* H *and* $\mathrm{G} \times \mathrm{H}$ *are groups, then* $(g_1 \star g_2, h_1 + h_2) = (g, h)$ *where* $g \in \mathrm{G}$*,* $h \in \mathrm{H}$ *and* $(g, h) \in \mathrm{G} \times \mathrm{H}$. ♣

Example 1.1 (Continued) Consider now the group $\mathbb{Z}_2 \times \mathbb{Z}_3$, this group is of order 6

$$\mathbb{Z}_2 \times \mathbb{Z}_3 = \{(\mathrm{e}, \mathrm{e}), (\mathrm{e}, \mathbf{r}_3), (\mathrm{e}, \mathbf{r}_3^2), (\mathbf{r}_2, \mathrm{e}), (\mathbf{r}_2, \mathbf{r}_3), (\mathbf{r}_2, \mathbf{r}_3^2)\}$$

The following Cayley table is obtained:

$\mathbb{Z}_2 \times \mathbb{Z}_3$	(e, e)	$(\mathrm{e}, \mathbf{r}_3)$	$(\mathrm{e}, \mathbf{r}_3^2)$	$(\mathbf{r}_2, \mathrm{e})$	$(\mathbf{r}_2, \mathbf{r}_3)$	$(\mathbf{r}_2, \mathbf{r}_3^2)$
(e, e)	(e, e)	$(\mathrm{e}, \mathbf{r}_3)$	$(\mathrm{e}, \mathbf{r}_3^2)$	$(\mathbf{r}_2, \mathrm{e})$	$(\mathbf{r}_2, \mathbf{r}_3)$	$(\mathbf{r}_2, \mathbf{r}_3^2)$
$(\mathrm{e}, \mathbf{r}_3)$	$(\mathrm{e}, \mathbf{r}_3)$	$(\mathrm{e}, \mathbf{r}_3^2)$	(e, e)	$(\mathbf{r}_2, \mathbf{r}_3)$	$(\mathbf{r}_2, \mathbf{r}_3^2)$	$(\mathbf{r}_2, \mathrm{e})$
$(\mathrm{e}, \mathbf{r}_3^2)$	$(\mathrm{e}, \mathbf{r}_3^2)$	(e, e)	$(\mathrm{e}, \mathbf{r}_3)$	$(\mathbf{r}_2, \mathbf{r}_3^2)$	$(\mathbf{r}_2, \mathrm{e})$	$(\mathbf{r}_2, \mathbf{r}_3)$
$(\mathbf{r}_2, \mathrm{e})$	$(\mathbf{r}_2, \mathrm{e})$	$(\mathbf{r}_2, \mathbf{r}_3)$	$(\mathbf{r}_2, \mathbf{r}_3^2)$	(e, e)	$(\mathrm{e}, \mathbf{r}_3)$	$(\mathrm{e}, \mathbf{r}_3^2)$
$(\mathbf{r}_2, \mathbf{r}_3)$	$(\mathbf{r}_2, \mathbf{r}_3)$	$(\mathbf{r}_2, \mathbf{r}_3^2)$	$(\mathbf{r}_2, \mathrm{e})$	$(\mathrm{e}, \mathbf{r}_3)$	$(\mathrm{e}, \mathbf{r}_3^2)$	(e, e)
$(\mathbf{r}_2, \mathbf{r}_3^2)$	$(\mathbf{r}_2, \mathbf{r}_3^2)$	$(\mathbf{r}_2, \mathrm{e})$	$(\mathbf{r}_2, \mathbf{r}_3)$	$(\mathrm{e}, \mathbf{r}_3^2)$	(e, e)	$(\mathrm{e}, \mathbf{r}_3)$

It can be observed that this table is symmetric. Hence the direct product group is Abelian. Let denote $\mathbf{r}_6 = (\mathbf{r}_2, \mathbf{r}_3)$.Then, the element $(\mathbf{r}_2, \mathbf{r}_3^2)$ appears to be the inverse of element $\mathbf{r}_6$ as read in the Cayley table. Additionally, one can see that $\mathbf{r}_6^6 = e$. As a consequence, the resulting direct product group is identified as $\mathbb{Z}_6 = \mathbb{Z}_3 \times \mathbb{Z}_2$.

In fact, we can note the following general property

Property *The direct product of Abelian groups is Abelian.*

We can see straight away that in order to generate non-commutative groups from commutative groups, something must be added to the notion of direct product. The notion of direct product can be generalised to the case where the group operation is more general. It is then called a semi-direct product.

Definition 1.18 (Semi-Direct Product)

Let G *and* H *be two groups and define the homomorphism* $\alpha : \mathrm{G} \longrightarrow \mathrm{Aut}(H)$*. The semi-direct product of* G *and* H*, written* $\mathrm{G} \ltimes \mathrm{H}$*, is defined as the group whose elements are* (g, h) *with* $g \in \mathrm{G}$ *and* $h \in \mathrm{H}$ *with the group* *operation:*

$$(g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1 \alpha_{g_1}(h_2)), \quad \forall g_1, g_2 \in \mathrm{G}, \forall h_1, h_2 \in \mathrm{H}$$

♣

Note *It is important to note that the semi-direct product is not unique as it depends on the choice of homomorphism α. Two different homomorphisms can lead to non-isomorphic semi-direct groups.*

Remark If one chooses homomorphism α to be the trivial homomorphism,

$$\forall g \in \mathrm{G}, \forall h \in \mathrm{H}\ \alpha_g(h) = h$$

sending every $g \in \mathrm{G}$ to the identity element of H then $\mathrm{G} \ltimes \mathrm{H} = \mathrm{G} \times \mathrm{H}$.

Example 1.9 An example of a semi-direct product of two groups is the Euclidian group $\mathrm{E}(2)$ of dimension 2. This group is composed of the translations, reflections and rotations of the plane. An operation $(\mathbf{q}, \mathbf{t}) \in \mathrm{E}(2)$ is composed of $\mathbf{q} \in \mathrm{O}(2)$, rotation about the origin or reflection about an axis passing through the origin, followed by $\mathbf{t} \in \mathbb{R}^2$ a translation. For a vector $\mathbf{u} \in \mathbb{R}^2$ the operation $(\mathbf{q}, \mathbf{t})$ transforms $\mathbf{u}$ to:

$$\mathbf{u}' = (\mathbf{q}, \mathbf{t})\mathbf{u} \equiv \mathbf{q}\mathbf{u} + \mathbf{t} \tag{1.1}$$

Furthermore, for $(\mathbf{q}_1, \mathbf{t}_1)(\mathbf{q}_2, \mathbf{t}_2) \in \mathrm{E}(2)$:

$$(\mathbf{q}_1, \mathbf{t}_1)(\mathbf{q}_2, \mathbf{t}_2)\mathbf{u} = \mathbf{q}_1\mathbf{q}_2\mathbf{u} + \mathbf{q}_1\mathbf{t}_2 + \mathbf{t}_1$$

This combination represents a first symmetry operation $(\mathbf{q}_2, \mathbf{t}_2)$ applied to vector $\mathbf{u}$ which created a vector $\mathbf{v}$, followed by a second operation $(\mathbf{q}_1, \mathbf{t}_1)$ applied to vector $\mathbf{v}$. The group $\mathrm{E}(2)$ is the semidirect product of $\mathrm{O}(2) \ltimes \mathbb{R}^2$. The homomorphism is $\alpha : \mathrm{O}(2) \longrightarrow \mathrm{Aut}(\mathbb{R}^2)$, with $\alpha_{\mathbf{q}}(\mathbf{t}) = \mathbf{q}\mathbf{t}$ and the group operation on $\mathbb{R}^2$ is addition. Thus, $(\mathbf{q}_1, \mathbf{t}_1)(\mathbf{q}_2, \mathbf{t}_2) = (\mathbf{q}_1\mathbf{q}_2, \mathbf{t}_1\alpha_{\mathbf{q}_1}(\mathbf{t}_2)) = (\mathbf{q}_1\mathbf{q}_2, \mathbf{t}_1 + \mathbf{q}_1\mathbf{t}_2)$

Example 1.1 (Continued) Consider now the group $\mathbb{Z}_2 \ltimes \mathbb{Z}_3$, this group is of order 6. The chosen automorphism for the semidirect product is the one that makes the non-identity element of $\mathbb{Z}_2$ send any element of $\mathbb{Z}_3$ to its inverse, that is: $\alpha_{e_{\mathbb{Z}_2}}(h) = h$ and $\alpha_{\mathbf{r}_2}(h) = h^{-1}$ for all $h \in \mathbb{Z}_3$. The group $\mathbb{Z}_2 \ltimes \mathbb{Z}_3$ is composed of the same 6 elements as the direct group $\mathbb{Z}_2 \times \mathbb{Z}_3$ but the Cayley table differs:

$\mathbb{Z}_2 \ltimes \mathbb{Z}_3$	(e, e)	$(\mathrm{e}, \mathbf{r}_3)$	$(\mathrm{e}, \mathbf{r}_3^2)$	$(\mathbf{r}_2, \mathrm{e})$	$(\mathbf{r}_2, \mathbf{r}_3)$	$(\mathbf{r}_2, \mathbf{r}_3^2)$
(e, e)	(e, e)	$(\mathrm{e}, \mathbf{r}_3)$	$(\mathrm{e}, \mathbf{r}_3^2)$	$(\mathbf{r}_2, \mathrm{e})$	$(\mathbf{r}_2, \mathbf{r}_3)$	$(\mathbf{r}_2, \mathbf{r}_3^2)$
$(\mathrm{e}, \mathbf{r}_3)$	$(\mathrm{e}, \mathbf{r}_3)$	$(\mathrm{e}, \mathbf{r}_3^2)$	(e, e)	$(\mathbf{r}_2, \mathbf{r}_3)$	$(\mathbf{r}_2, \mathbf{r}_3^2)$	$(\mathbf{r}_2, \mathrm{e})$
$(\mathrm{e}, \mathbf{r}_3^2)$	$(\mathrm{e}, \mathbf{r}_3^2)$	(e, e)	$(\mathrm{e}, \mathbf{r}_3)$	$(\mathbf{r}_2, \mathbf{r}_3^2)$	$(\mathbf{r}_2, \mathrm{e})$	$(\mathbf{r}_2, \mathbf{r}_3)$
$(\mathbf{r}_2, \mathrm{e})$	$(\mathbf{r}_2, \mathrm{e})$	$(\mathbf{r}_2, \mathbf{r}_3^2)$	$(\mathbf{r}_2, \mathbf{r}_3)$	(e, e)	$(\mathrm{e}, \mathbf{r}_3^2)$	$(\mathrm{e}, \mathbf{r}_3)$
$(\mathbf{r}_2, \mathbf{r}_3)$	$(\mathbf{r}_2, \mathbf{r}_3)$	$(\mathbf{r}_2, \mathrm{e})$	$(\mathbf{r}_2, \mathbf{r}_3^2)$	$(\mathrm{e}, \mathbf{r}_3)$	(e, e)	$(\mathrm{e}, \mathbf{r}_3^2)$
$(\mathbf{r}_2, \mathbf{r}_3^2)$	$(\mathbf{r}_2, \mathbf{r}_3^2)$	$(\mathbf{r}_2, \mathbf{r}_3)$	$(\mathbf{r}_2, \mathrm{e})$	$(\mathrm{e}, \mathbf{r}_3^2)$	$(\mathrm{e}, \mathbf{r}_3)$	(e, e)

This Cayley table is not symmetric anymore. Let denote $\boldsymbol{\pi} = (\mathbf{r}_2, \mathbf{r}_3)$. It can be observed that the inverse of this element is itself: $\boldsymbol{\pi}^2 = e$. As a consequence, one can identify: $\mathbb{D}_3 = \mathbb{Z}_2 \ltimes \mathbb{Z}_3$.

2 Lattice of groups

The different subgroups of a given group have various relationships with each other, some are included in others, others share only the neutral element. We will introduce here a tool that will underline these relationships. A **partial order relationship** can be defined for groups and classes of subgroups, to formalise the way in which these subgroups are organised. This provides a graphical representation of these connections. This representation is used in elasticity to order symmetry classes, and has an even more important role in structural stability because it indicates possible symmetry transitions at bifurcations.

Let's start by introducing the definition of a lattice on an abstract structure ruled with partial ordering:

Definition 1.19 (Lattice)

A lattice is an algebraic structure consisting of a set of elements $\mathcal{L} = \{A, B, \ldots, Z\}$, and a partial ordering relation, denoted $\leq$, satisfying the following properties for $A, B, C \in \mathcal{L}$:

- *Reflexivity*
$$\forall A \in \mathcal{L}, \quad A \leq A$$
- *Transitivity*
$$\text{if } A \leq B \text{ and } B \leq C, \quad \text{then } A \leq C$$
- *Antisymmetry*
$$\text{if } A \leq B \text{ and } B \leq A, \quad \text{then } A = B$$

♣

Remark A lattice is a partially ordered set (**poset**), meaning some of its elements may not be comparable ($a \nleq b$ and $b \nleq a$).

Partial ordering can be represented by a **tree diagram** where lines indicate partial relationships.

Example 1.10 In the case of the example lattice tree diagram presented Figure 1.5, the following ordering relations hold $\mathrm{H} \leq E$ and $\mathrm{H} \leq \mathrm{G}$ but E and G are not comparable.

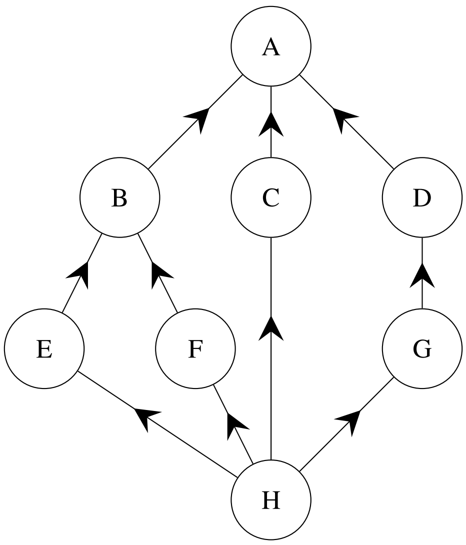


Figure 1.5: Example of lattice tree diagram

2.1 Lattice of subgroups

The notion of lattice can be applied to group-subgroup relations. In this case, partial ordering can be represented by a tree diagram where lines only join maximal subgroups and minimal supergroups. This type of partial ordering relation defines a **lattice of subgroups**.

Definition 1.20 (Maximal subgroup)

A maximal subgroup H *of* G*, is a subgroup* $\mathrm{H} \subset \mathrm{G}$ *such that there does not exist* $\mathrm{K} \subset \mathrm{G}$ *such that*
$$\mathrm{H} \subset \mathrm{K} \subset \mathrm{G}$$

♣

Definition 1.21 (Minimal supergroup)

A minimal supergroup H *of* I *of* G*, is a supergroup* H *such that* $\mathrm{I} \subset \mathrm{H}$ *and for which there does not exists* K*,* $\mathrm{I} \subset \mathrm{K}$ *such that*

$$\mathrm{I} \subset \mathrm{K} \subset \mathrm{H}$$

♣

Example 1.1 (Continued) Back to Example 1.1 on the symmetry group D_3 of the equilateral triangle in plane, the maximal subgroups of D_3 are $\mathrm{D}_1^{(1)}, \mathrm{D}_1^{(2)}, \mathrm{D}_1^{(3)}, \mathrm{Z}_3$. The minimal supergroups of $1 \subset \mathrm{D}_3$ are the same groups: $\mathrm{D}_1^{(1)}, \mathrm{D}_1^{(2)}, \mathrm{D}_1^{(3)}, \mathrm{Z}_3$. This is a degenerate case where maximal subgroups and minimal supergroups are the same. This is due to the fact that D_3 is a group of low order.
If one considers a bigger group such as D_6 for instance, then its maximal subgroups are $\mathrm{D}_3,\ (\mathrm{D}_2^{(i)}\ i = 1...6),\ \mathrm{Z}_6$ whereas the minimal supergroups of $1 \subset \mathrm{D}_6$ are $\mathrm{Z}_2,\ \mathrm{Z}_3,\ (\mathrm{D}_1^{(i)}\ i = 1...6)$.
The lattice tree diagram of group D_3 is presented Figure 1.6.

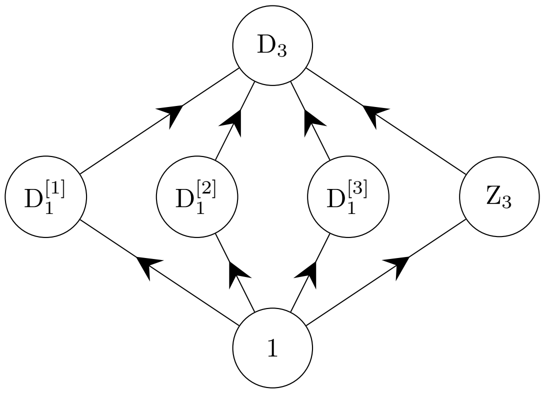


Figure 1.6: Lattice tree diagram of group D_3

2.2 Lattice of subgroup-classes

The same approach can be done with subgroup-classes instead of groups. However, we need another definition for the partial order relationship since it would then have to be defined up to conjugacy relations.

Definition 1.22 (Partial ordering for classes)

Denote $\mathfrak{I} = \{[1], \ldots, [\mathrm{G}]\}$ *as the set of all subgroup classes of* G*. Then, we can construct a subgroup-class lattice when the partial ordering is given*

$$[\mathrm{H}] \leq [\mathrm{K}], \quad \textit{if } g\mathrm{H}g^{-1} \subset \mathrm{K}$$

for some $g \in \mathrm{G}$.

♣

Example 1.1 (Continued) For D_3, $\mathfrak{I} = \{[1], [\mathrm{D}_1], [\mathrm{Z}_3], [\mathrm{D}_3]\}$ with

- $[\mathrm{D}_3] = \{\mathrm{D}_3\}$
- $[\mathrm{Z}_3] = \{\mathrm{Z}_3\}$
- $[\mathrm{D}_1] = \{\mathrm{D}_1^{(1)}, \mathrm{D}_1^{(3)}, \mathrm{D}_1^{(3)}\}$
- $[1] = \{1\}$

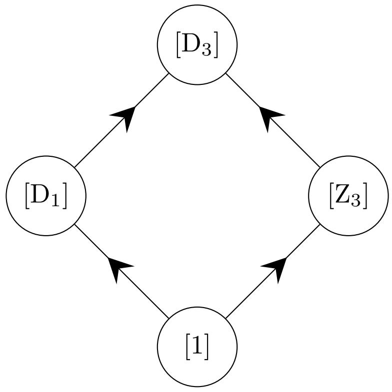
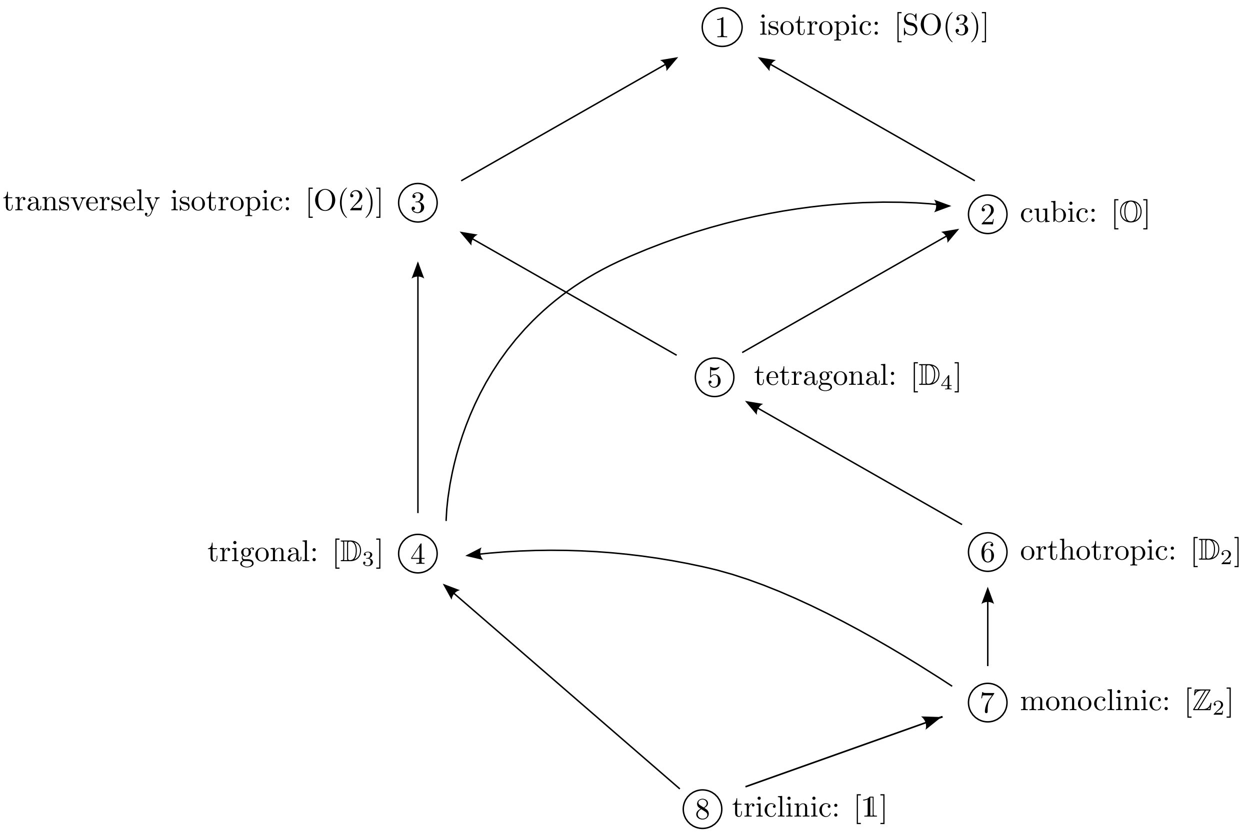


Figure 1.7: Lattice of subgroup-classes of $[\mathrm{D}_3]$

Example 1.11 For a more complex example, we can represent the poset of 3D elasticity symmetry classes [15].



Chapter 2 Fundamentals of Group Representations

Content

- *Preliminaries*
- *Composition, decomposition and irreducibility*
- *Character theory*
- *Extension to the orthogonal group*

In the mathematical field of representation theory, group representations describe abstract groups in terms of bijective linear transformations of a vector space $\mathbb{V}$ to itself; in particular, they can be used to represent group elements as invertible matrices so that the group operation is represented by matrix multiplication.

Representations of groups are important in physics because, for example, they describe how the symmetry group of a physical system affects the solutions of equations describing that system. Even when the studied system is nonlinear, as in equivariant bifurcation theory [1].

In mechanics, this theory can be applied to the study of linear constitutive laws (material approach) [4, 15] and to the study of the stability of symmetrical structures (structure approach). These situations will differ essentially in the choice of the vector space representing the physics of the system under study:

Material case the vector space $\mathbb{V}$ will be a tensor space of order n constructed from $\mathbb{R}^d$, where d is the dimension of the physical space under consideration;

Structural case the vector space will be the configuration space of the structure under study and will have the form of a Cartesian product of copies of $\mathbb{R}^n$ with n the number of degrees of freedom of the model under study.

Before getting to the heart of the matter, and given their importance for the rest of this section, we'll start with a few reminders about matrix groups.

Some classical references concerning this chapter are the following monographies:

- Baker, A. (2003). Matrix groups: An introduction to Lie group theory. Springer Science;
- Fulton, W., Harris, J. (2013). Representation theory: a first course (Vol. 129). Springer Science ;
- Serre, J. P. (1971). Représentation linéaire des groupes finis. Hermann, Paris.

1 Preliminaries

1.1 Matrix Group

Let $\mathrm{M}_{m,n}(\mathbb{k})$ be the set of $m \times n$ matrices with entries in $\mathbb{k}$, here $\mathbb{k}$ is either $\mathbb{R}$ or $\mathbb{C}$. We set $\mathrm{M}_n(\mathbb{k}) = \mathrm{M}_{n,n}(\mathbb{k})$, the set of square matrices of dimension n.

Recall the determinant function $\det : \mathrm{M}_n(\mathbb{k}) \to \mathbb{k}$.

Theorem 2.1

$\det : \mathrm{M}_n(\mathbb{k}) \to \mathbb{k}$ *has the following properties:*

- *For* $\mathrm{A},\ \mathrm{B} \in \mathrm{M}_n(\mathbb{k}),\ \det(\mathrm{AB}) = \det(\mathrm{A})\det(\mathrm{B})$*;*
- $\mathrm{A} \in \mathrm{M}_n(\mathbb{k})$ *is invertible if and only if* $\det(\mathrm{A}) \neq 0$*;*
- $\det(\mathrm{I}_n) = 1$.

♡

We use the notation

$$\mathrm{GL}(n, \mathbb{k}) = \{\mathrm{M} \in \mathrm{M}_n(\mathbb{k}) | \det \mathrm{M} \neq 0\}$$

for the set of invertible $n \times n$ matrices. $\mathrm{GL}(n, \mathbb{k})$ is called the $n \times n$ **general linear group**. The following theorem states that this set is indeed a group for the matrix multiplication.

Theorem 2.2

The set $\mathrm{GL}(n, \mathbb{k})$ *is a group under matrix multiplication.* ♡

Note *The group* $\mathrm{GL}(n, \mathbb{k})$ *is unbounded, hence it is not a compact group.*

Remark In solid mechanics, the group $\mathrm{GL}(d, \mathbb{R})$ is used to define the group of homogeneous transformations [14]

From the general linear group, it is possible to define other matrix groups.

Definition 2.1 (Matrix Group)

A subgroup $\mathrm{G} \leq \mathrm{GL}(n, \mathbb{k})$ *which is also a closed subspace is called a* ***matrix group*** *over* $\mathbb{k}$. ♣

Among the various groups of matrices, the orthogonal group and the unitary group will be of key importance for our subject.

Definition 2.2 (Orthogonal group)

The following set of matrices

$$\mathrm{O}(d) = \{\mathrm{Q} \in \mathrm{GL}(d, \mathbb{R}) | \mathrm{Q}^T \mathrm{Q} = \mathrm{I}_d\}$$

in which Q^T *is the transpose of* Q *is called the real orthogonal group. Elements of* $\mathrm{O}(d)$ *are called* ***orthogonal matrices****.* ♣

The set of matrices encoding rotations of vectors in $\mathbb{R}^d$ constitute a subgroup of $\mathrm{O}(d)$, denoted $\mathrm{SO}(2)$ and called the **special orthogonal group**.

Definition 2.3 (Unitary group)

The following set of matrices

$$\mathrm{U}(d) = \{\mathrm{U} \in \mathrm{GL}(d, \mathbb{C}) | \overline{\mathrm{U}}^T \mathrm{U} = \mathrm{I}_d\}$$

in which $\overline{\mathrm{U}}^T$ *is the conjugate transpose of* U *is called the unitary group. Elements of* $\mathrm{U}(d)$ *are called* ***unitary matrices****.* ♣

The group $\mathrm{U}(1)$ is the set of complex numbers $z \in \mathbb{C}$ satisfying, $\overline{z}z = 1$, ie. the set of complex number of module 1. Hence elements of $\mathrm{U}(1)$ can be parameterised as follows $z = \mathrm{e}^{i\theta}$, $\theta \in \mathbb{R}$. The group $\mathrm{U}(1)$ is isomorphic to $\mathrm{SO}(2)$, the special orthogonal group in two dimensions.

1.2 Basic definitions

Definition 2.4 (Group representation)

Let G *be any group and* $\mathbb{V}$ *a* $\mathbb{k}$*-vector space. A representation* ρ *of* G *on* $\mathbb{V}$ *is a morphism from* G *into the general linear group* $\mathrm{GL}(\mathbb{V})$*:*

$$\rho : \mathrm{G} \to \mathrm{GL}(\mathbb{V})$$

that preserves the group product, i.e. such that for each element $g \in \mathrm{G}$ *we have*

- $\rho(g_1 g_2) = \rho(g_1)\,\rho(g_2)$;
- $\rho(g^{-1}) = \rho(g)^{-1}$;
- $\rho(e) = \mathrm{I}_{\mathbb{V}}$.

♣

Remark Sometimes the group representation is denoted by $(\rho, \mathbb{V}, \mathrm{G})$. When there is no ambiguity about G, or when the group is included in the name of the representation (e.g. the representation is called G-representation), we simply note $(\rho, \mathbb{V})$. When the morphism is clear from the context, we sometimes simply refer to the representation of G as $\mathbb{V}$. This is a fairly classic abuse of definition.

The dimension of a representation $(\rho, \mathbb{V})$ is the dimension of the support space $\mathbb{V}$, it is sometimes also called the degree of ρ.

Note *The choice of a basis for $\mathbb{k}$-vector space $\mathbb{V}$ implies the classical isomorphism $\mathbb{V} \simeq \mathbb{k}^n$ with $n = \dim \mathbb{V}$. As a consequence $\mathrm{GL}(\mathbb{V}) \simeq \mathrm{GL}(n, \mathbb{k})$ meaning that group elements are described by invertible $n \times n$ matrices. When the choice of the field is understood from the context, it will be omitted.*

Example 2.11 A representation of degree 1 of a group G is a group homomorphism $\rho : \mathrm{G} \to \mathbb{C}^\star$, where $\mathbb{C}^\star$ denotes the multipmicative group of nonzero complex numbers. Since each element of G are of finite order, the value $\rho(g)$ are roots of unity.

Example 2.12 There exists a trivial unidimensionnal representation called the **identity representation** which is the G-representation $(\rho^{(e)}, \mathbb{k})$

$$\forall g \in \mathrm{G}, \rho^{(e)}(g) = 1$$

The kernel of the representation $\ker \rho^{(e)}$ is in this case the whole group G. The image of $\rho^{(e)}$ is isomorphic to the quotient group[a] $\mathrm{G}/\ker \rho^{(e)} = 1$.

[a] This result is known as the **first isomorphism theorem**.

Example 2.13 $\mathbb{E}\mathrm{la}$, the vector space of elasticity tensors has, in $\mathbb{R}^3$, the following structure

$$\mathbb{E}\mathrm{la} \simeq S^2(S^2(\mathbb{R}^3))$$

in which $S^2(\mathbb{V})$ stands for the symmetrised tensor product. This is a space of dimension 21, so the elasticity tensors must be considered as **vectors of dimension 21**. And the transformation matrices are hence square matrices of dimension 21×21.
This description is complementary to that of a 4th-order tensor, i.e. an object with 4 indices. In this latter view, transformations will be tensors of order 8 which act on 4th-order tensors instead of square matrices of dimension 21×21 acting on vectors of dimension 21.

Example 2.1 (Continued) Back to the example of the symmetry group of an equilateral triangle, the following mapping $(\rho^{(1)}, \mathbb{R}, \mathrm{D}_3)$ is a one-dimensional representation of D_3 on $\mathbb{R}$:

D_3	e	$\mathbf{r}_3$	$\mathbf{r}_3^2$	$\boldsymbol{\pi}$	$\mathbf{r}_3\boldsymbol{\pi}$	$\mathbf{r}_3^2\boldsymbol{\pi}$
$\rho^{(1)}(g)$	1	1	1	-1	-1	-1

The kernel of the representation $\ker \rho^{(1)}$ is here the group Z_3. The image of $\rho^{(1)}$ is isomorphic to the quotient group $\mathrm{D}_3/\mathrm{Z}_3 = \mathrm{Z}_2$. One can easily check that this representation preserves the Cayley table of the group D_3 presented in Table 1.1. This representation is known as the **sign representation**.
Another classical representation for D_3 is the standard action on vectors of $\mathbb{R}^3$: $(\rho^{(3)}, \mathbb{R}^3)$. With respect to a direct orthonormal basis such as $\underline{e}_3$ corresponds to the axis of the rotations, we have

D_3	e	$\mathbf{r}_3$	$\mathbf{r}_3^2$	$\boldsymbol{\pi}$	$\mathbf{r}_3\boldsymbol{\pi}$	$\mathbf{r}_3^2\boldsymbol{\pi}$
$\rho^{(3)}(g)$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}-1/2&-\sqrt{3}/2&0\\\sqrt{3}/2&-1/2&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}-1/2&\sqrt{3}/2&0\\-\sqrt{3}/2&-1/2&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}-1/2&-\sqrt{3}/2&0\\-\sqrt{3}/2&1/2&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}-1/2&\sqrt{3}/2&0\\\sqrt{3}/2&1/2&0\\0&0&1\end{pmatrix}$

Consider two representations $(\rho^{(\alpha)}, \mathbb{V}_\alpha)$ and $(\rho^{(\beta)}, \mathbb{V}_\beta)$ of a same group G. These representations are said to be equivalent (or isomorphic) if there exists a linear isomorphism $T : \mathbb{V}_\alpha \to \mathbb{V}_\beta$ which "transforms" $\rho^{(\alpha)}$ into $\rho^{(\beta)}$, that is, which satisfies the identity

$$\forall g \in \mathrm{G},\ \rho^{(\beta)}(g)T = T\rho^{(\alpha)}(g)$$

More formally Such a bijective linear map T is then said to be **commutative**. This property is also called interwinning or **equivariant** and can be represented by the following commutative diagram for any $g \in \mathrm{G}$:

$$\begin{tikzcd}
\mathbb{V}_\alpha \arrow[r, "\mathbf{T}"] \arrow[d, "\rho^{(\alpha)}(g)"'] & \mathbb{V}_\beta \arrow[d, "\rho^{(\beta)}(g)"] \\
\mathbb{V}_\alpha \arrow[r, "\mathbf{T}"] & \mathbb{V}_\beta
\end{tikzcd}$$

When $\rho^{(\alpha)}$ and $\rho^{(\beta)}$ are given in matrix form, this means that there exists an invertible matrix T such that

$$\forall g \in \mathrm{G},\ \rho^{(\beta)}(g) = T\rho^{(\alpha)}(g)T^{-1}$$

The equivalence can therefore be understood as a change of basis which links the set of matrices $\rho^{(\alpha)}(g)$ to the set $\rho^{(\beta)}(g)$.

It is interesting to define representations that preserve the scalar product.

Definition 2.5 (Unitary representation)

Let $< \cdot, \cdot >$ be a scalar product on $\mathbb{V}$. A G-representation $(\rho, \mathbb{V})$ is unitary if and only if it preserves the scalar product, that is:

$$\forall g \in \mathrm{G},\ \forall x, y \in \mathbb{V},\ \langle \rho(g)x, \rho(g)y \rangle =< x, y >$$

♣

These representations, known as **unitary**, are defined from unitary matrices.

Property *A representation of dimension n is said to be unitary if*

$$\forall g \in \mathrm{G}, \quad \rho(g) \in \mathrm{U}(n)$$

Note *If $\mathbb{V}$ is a $\mathbb{R}$-vector space, the representation is said to be **orthogonal** instead of unitary, and the classical transpose is used instead of the Hermitian one.*

Given any Hermitian product, it is always possible to construct an unitary one by averaging it over the group elements, this is the content of the following theorem:

Theorem 2.3

Every representation of a finite group is equivalent to a unitary representation.

♡

In this respect, it is not restrictive to assume that all our representations will be considered as unitary or orthogonal, depending on the situation.

2 Composition, decomposition and irreducibility

2.1 Composition of representations

Representations can be combined to produce more complex representations. There are two basic mechanisms for doing this:

- direct sum $\oplus$;
- tensor product $\otimes$.

This approach can obviously be seen in reverse, and we can wonder how a given representation decomposes into a collection of simpler elements. We will return to this question, which is of fundamental interest, shortly.

Definition 2.6 (Direct sum)

Let two G*-representations* $(\rho^{(\alpha)}, \mathbb{V}_\alpha)$ *and* $(\rho^{(\beta)}, \mathbb{V}_\beta)$ *with* $\dim \mathbb{V}_\alpha = n_\alpha$ *and* $\dim \mathbb{V}_\beta = n_\beta$*. The direct sum of these representations, denoted* $\rho^{(\alpha)} \oplus \rho^{(\beta)}$*, is a representation* $(\rho^{(\alpha)} \oplus \rho^{(\beta)}, \mathbb{V}_\alpha \oplus \mathbb{V}_\beta)$ *of dimension* $n_\alpha + n_\beta$ *such that* $\left(\rho^{(\alpha)} \oplus \rho^{(\beta)}\right)(g) = \left[\begin{array}{c|c} \rho^{(\alpha)}(g) & 0 \\ \hline 0 & \rho^{(\beta)}(g) \end{array}\right]$ ♣

Example 2.1 (Continued) Let's go back to example Table 2.1. Note that the transformation matrices are block diagonal. This indicates that the representation is the direct sum of a 2D representation and a 1D one, i.e. $\mathbb{V} = \mathbb{R}^2 \oplus \mathbb{R}$. More precisely, $(\rho^{(3)}, \mathbb{R}^3) = (\rho^{(2)} \oplus \rho^{(e)}, \mathbb{R}^2 \oplus \mathbb{R})$ with:

D_3	e	$\mathbf{r}_3$	$\mathbf{r}_3^2$	$\boldsymbol{\pi}$	$\mathbf{r}_3\boldsymbol{\pi}$	$\mathbf{r}_3^2\boldsymbol{\pi}$
$\rho^{(e)}(g)$	1	1	1	1	1	1
$\rho^{(2)}(g)$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$	$\begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}$	$\begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$

The form of $\rho^{(2)}(\mathbf{r}_3)$ indicates that the decomposition cannot be continued, at least in $\mathbb{R}$. The representation $\rho^{(2)}$ is sometimes called **the standard representation**. This representation is faithful since $\ker \rho^{(2)} = 1$.

Definition 2.7 (Tensor product)

Let two G*-representations* $(\rho^{(\alpha)}, \mathbb{V}_\alpha)$ *and* $(\rho^{(\beta)}, \mathbb{V}_\beta)$ *with* $\dim \mathbb{V}_\alpha = n_\alpha$ *and* $\dim \mathbb{V}_\beta = n_\beta$*. The tensor product of these representations, denoted* $\rho^{(\alpha)} \otimes \rho^{(\beta)}$*, is a representation* $(\rho^{(\alpha)} \otimes \rho^{(\beta)}, \mathbb{V}_\alpha \otimes \mathbb{V}_\beta)$ *of dimension* $n_\alpha n_\beta$ *such that, in index notations, one has*

$$\forall g \in \mathrm{G}, \quad \left(\rho^{(\alpha)} \otimes \rho^{(\beta)}\right)_{iJkL}(g) = \rho^{(\alpha)}_{ik}(g)\rho^{(\beta)}_{JL}(g), \quad 1 \leq i,k \leq n_\alpha,\ 1 \leq J,L \leq n_\beta$$

♣

Example 2.1 (Continued) Consider the standard representation $(\rho^{(2)}, \mathbb{R}^2, \mathrm{D}_3)$ as defined above Let $\mathcal{B} = \{\underline{e}_1, \underline{e}_2\}$ be an orthonormal basis of $\mathbb{R}^2$, and consider the space of second-order tensors $\mathbb{V} = \mathbb{R}^2 \otimes \mathbb{R}^2$ with elements $\underset{\sim}{A} = A_{ij}\underline{e}_i \otimes \underline{e}_j$. The transformation of $\underset{\sim}{A}$ by g is a new tensor (active point of view) $\underset{\sim}{A}^\star = A^\star_{ij}\underline{e}_i \otimes \underline{e}_j$, with

$$A^\star_{ij} = \varrho_{ijkl}(g)A_{kl} = \rho^{(2)}_{ik}(g)\rho^{(2)}_{jl}(g)A_{kl}$$

Note that this expression corresponds to the well-know transformation rule:

$$\underset{\sim}{A}^\star = \underset{\sim}{Q}\underset{\sim}{A}\underset{\sim}{Q}^T$$

Thanks to the tensor product, we have built a representation of D_3 on the space of second-order tensors: $\left(\varrho = \rho^{(2)} \otimes \rho^{(2)}, \mathbb{R}^2 \otimes \mathbb{R}^2\right)$.

Now let's consider the following orthonormal basis for $\mathbb{V}$

$$\underline{E}_1 = \underline{e}_1 \otimes \underline{e}_1,\ \underline{E}_2 = \underline{e}_2 \otimes \underline{e}_2,\ \underline{E}_3 = \underline{e}_1 \otimes \underline{e}_2,\ \underline{E}_4 = \underline{e}_2 \otimes \underline{e}_1$$

with respect to this basis we have the following matrix representation of the element $\mathbf{r}_3$

$$[\varrho(\mathbf{r}_3)] = \begin{pmatrix} 1/4 & 3/4 & \sqrt{3}/4 & \sqrt{3}/4 \\ 3/4 & 1/4 & -\sqrt{3}/4 & -\sqrt{3}/4 \\ -\sqrt{3}/4 & \sqrt{3}/4 & 1/4 & -3/4 \\ -\sqrt{3}/4 & \sqrt{3}/4 & -3/4 & 1/4 \end{pmatrix}$$

Note *It is common to define the symmetrised tensor product in the case where both representations are identical and act* *on a vector space $\mathbb{V}$ of dimension n. This symmetrised tensor product, denoted $\otimes^s$, is defined, in index notations, as:*

$$(\rho \otimes^s \rho)_{ijkl} = \frac{1}{2}\left(\rho_{ik}\rho_{jl} + \rho_{il}\rho_{jk}\right), \quad 1 \leq i,j,k,l \leq n$$

where the explicit dependance in the element $g \in \mathrm{G}$ has been omitted for the sake of simplicity.

Example 2.1 (Continued) Consider the space of symmetric second-order tensors $\mathbb{V} = S^2(\mathbb{R}^2)$ which is of dimension 3. The transformation of $\underset{\sim}{\mathrm{S}}$ by g is a new tensor $\underset{\sim}{\mathrm{S}}^\star = S^\star_{ij}\underline{\mathrm{e}}_i \otimes \underline{\mathrm{e}}_j$, with

$$S^\star_{ij} = \varrho_{ijkl}(g)S_{kl} = \left(\rho^{(2)} \otimes^s \rho^{(2)}\right)_{ijkl} S_{kl}$$

Let define the following orthonormal basis for $\mathbb{V}$:

$$\underline{\mathrm{E}}_1 = \underline{\mathrm{e}}_1 \otimes \underline{\mathrm{e}}_1,\ \underline{\mathrm{E}}_2 = \underline{\mathrm{e}}_2 \otimes \underline{\mathrm{e}}_2,\ \underline{\mathrm{E}}_3 = \frac{\sqrt{2}}{2}\left(\underline{\mathrm{e}}_1 \otimes \underline{\mathrm{e}}_2 + \underline{\mathrm{e}}_2 \otimes \underline{\mathrm{e}}_1\right)$$

This correspond to the well-known Kelvin representation of elasticity tensors. With respect to this basis we have the following matrix representation of the element $\mathbf{r}_3$

$$[\varrho(\mathbf{r}_3)] = \begin{pmatrix} 1/4 & 3/4 & \sqrt{6}/4 \\ 3/4 & 1/4 & -\sqrt{6}/4 \\ -\sqrt{6}/4 & \sqrt{6}/4 & 1/2 \end{pmatrix}$$

2.2 Irreducible representations and Schur's Lemma

As mentioned above, we can reverse the approach and try to decompose a given representation into a direct sum of more elementary representations, until we reach an elementary level for which we can go no further. The representations appearing in this maximal decomposition are said to be **irreducible**. Irreducible representations are the building blocks from which all other representations can be constructed. In some ways, they are the equivalent of the prime numbers for representation theory. An important result of the theory of finite groups is that, for a given group, there is a finite number of these elementary representations. What's more, they are known for a large number of classical groups.

Decomposing a representation consists in identifying subspaces that are invariant under the action of G.

Definition 2.8 (Invariant subspace)

Let $(\rho, \mathbb{V})$ be a representation of G. A vector subspace $\mathbb{W} \subset \mathbb{V}$ is called invariant under ρ (or under G, if the name of the representation is understood) if

$$\forall g \in \mathrm{G}, \rho(g)\mathbb{W} = \mathbb{W}.$$

♣

If $\mathbb{W} \subset \mathbb{V}$ is G-stable, then

$$\mathbb{V} = \mathbb{W} \oplus \mathbb{W}^\perp$$

in which, the complementary space is also G-stable. This process can be iterated until an ultimate step, for which the representation can no longer be decomposed

Definition 2.9 (Irreducible representation)

*A G-representation $(\mathbb{W}, \rho)$ is called irreducible if $\mathbb{W} \neq 0$ and if the only vector subspaces of $\mathbb{W}$ invariant under ρ are 0 and $>$ itself. Such a subspace $\mathbb{W}$ is said to be a G-**irreducible space**.*

♣

Remark Every one-dimensional representation is irreducible.

Note *Irreducibility generally depends on the field $\mathbb{C}$ or $\mathbb{R}$.*

- *A representation that is irreducible over $\mathbb{C}$ is called **absolutely irreducible**;*
- *A representation that is irreducible over $\mathbb{R}$ might not be irreducible over $\mathbb{C}$. These are called **physically irreducible** representations in some physical communities [25]. A physically irreducible representation is actually reducible*

over $\mathbb{C}$ and is equivalent to the direct sum of a complex irreducible representation and its complex conjugate.

Example 2.1 (Continued) The representation $(\rho^{(2)}, \mathbb{R}^2, \mathrm{D}_3)$ is irreducible over $\mathbb{R}$ but not over $\mathbb{C}$ since it can be reduced to $\begin{pmatrix} \mathrm{e}^{i\theta} & 0 \\ 0 & \mathrm{e}^{-i\theta} \end{pmatrix}$ on $\mathbb{C}$ where $\theta = \frac{2\pi}{3}$.

Property

- *All groups* G *have, as an irreducible representation, the* ***trivial representation*** *which maps all elements of* G *to the identity element.*
- *All irreducible representations of a finite group are of finite dimension, the maximum degree being the order of the group.*

We also have the following result which indicates, for each group, the number of its irreducible representations

Theorem 2.4

The number of irreducible representations in a group is equal to the number of classes of this group. ♡

A natural question now is how to determine the irreducible representations of a given finite group. We will not deal with this question here, as the results for the situations we are interested in are tabulated in numerous references [10, 28, 29]. The situations concern elementary groups, and then there are techniques to determine irreducible representations of any finite group by computational procedures either by restriction or induction methods from irreducible representations of either larger or smaller groups, respectively. These procedures have been coded and optimised into computational procedures [5, 13, 24].

In the majority of situations that will interest us in mechanics, the groups considered will be finite subgroups of $\mathrm{O}(3)$. In reference to crystallography, we will refer to these as "point groups". On a personal level, when we are looking for information on the irreps of these groups in our work, we consult the following sites : [8, 17]. When dealing with standard point groups in the context of physical chemistry and quantum mechanics, irreducible representation have standardised names [21, 25] One dimensional irreducible representations are either denoted by A_i or B_i. For representations of type A_i the value of the principal generator is 1, while it is -1 for representation of type B_i. Subscripts 1 and 2 corresponds to 1 and -1 associated to the secondary generator. The A_i representations are classical and correspond to the **trivial representation** for A_1, and to the **sign representation** for A_2 [16]. Two dimensional representations are denoted by E_i, while three dimensional representations are indicated by T_i. We can note the representations G and H of dimensions 4 and 5 respectively which are associated with the groups of the icosahedron and which play an important role in the study of the properties of quasi-icosahedral crystals [12].

As the context of this book is more general than that used to name these irreducible representations, the choice has been made to stick to the abstract notation $\rho^{(\alpha)}$ where $\rho^{(1)}$ is the trivial representation equivalent to A_1 in the notation above.

Theorem 2.5 (Schur's Lemma)

Let $(\rho^{(\alpha)}, \mathbb{V}_\alpha)$ and $(\rho^{(2)}, \mathbb{V}_\beta)$ be two irreducible G*-representations over $\mathbb{C}$. Consider ϕ be an equivariant linear map between $(\rho^{(\alpha)}, \mathbb{V}_\alpha)$ and $(\rho^{(\beta)}, \mathbb{V}_\beta)$.*

- *if $\rho^{(\alpha)}$ and $\rho^{(\beta)}$ are not equivalent, then $\phi = 0$;*
- *if $\mathbb{V}_\alpha = \mathbb{V}_\beta = \mathbb{V}$ and $\rho^{(\alpha)} = \rho^{(\beta)} = \rho$, then ϕ is a scalar multiple of the identity of $\mathbb{V}$.* ♡

The Schur's Lemma means that the only non trivial matrices that commute with all the matrices of an **absolutely irreducible** representation are scalar multiples of the identity. It is important to stress that this lemma, as formulated above and encountered in numerous references, depends crucially on the fact that we are working on $\mathbb{C}$.

On $\mathbb{R}$, this lemma should be adapted

Theorem 2.6 (Real version of Schur's Lemma)

If ρ is irreducible over $\mathbb{R}$ but not $\mathbb{C}$, then the second point of Schur's lemma transforms into $\phi = \left[\begin{array}{c|c} a\underset{\sim}{\mathrm{I}} & b\underset{\sim}{\mathrm{I}} \\ \hline -b\underset{\sim}{\mathrm{I}} & a\underset{\sim}{\mathrm{I}} \end{array}\right]$ *for some $a, b \in \mathbb{R}$, where $\underset{\sim}{\mathrm{I}}$ is the identity matrix of dimension $\frac{d}{2}$ with d the dimension of $\mathbb{V}$.* ♡

this point is important when working with **physically irreducible representations**, i.e. representations that are irreducible over $\mathbb{R}$ but not $\mathbb{C}$.

2.3 Reducible representations

In this section the general structure of a reducible representation is expressed as a direct sum of irreducible representations.

Definition 2.10 (Reducible representations)

A representation of a group G is said to be ***reducible*** *if it is not irreducible.* ♣

The following result ensures that this process can be carried out

Theorem 2.7 (Maschke's theorem)

Every finite dimensional G-representation $(\rho, \mathbb{V})$ is a direct sum of irreducible representations. ♡

In practice as soon as there is a non trivial G-invariant subspace, the representation is reducible.

Property *A G-representation $(\rho, \mathbb{V})$ is reducible (over $\mathbb{C}$ or $\mathbb{R}$) if it is equivalent to a G-representation $(\varrho, \mathbb{W})$ that has the form*

$$\forall g \in \mathrm{G}, \quad \begin{pmatrix} [\rho^{(\alpha)}(g)] & 0 \\ 0 & [\rho^{(\beta)}(g)] \end{pmatrix}, \quad \mathbb{W} = \mathbb{V}_\alpha \oplus \mathbb{V}_\beta$$

The previous step can be iterated until the obtained representations become irreducible. In this decomposition into direct sum, the same irreducible representation may appear several times. The multiplicity of an irreducible representation $\rho^{(\alpha)}$ in the decomposition is noted m_α. Let $(\rho, \mathbb{V})$ be a G-representation and let

$$\rho = \bigoplus_{\alpha=1}^{N} m_\alpha \rho^{(\alpha)}$$

be the decomposition of ρ into blocks $m_\alpha \rho^{(\alpha)}$ called **isotypic components**. The support of the isotypic component $m_\alpha \rho^{(\alpha)}$, is

$$m_\alpha \mathbb{V}_\alpha = \underbrace{\mathbb{V}_\alpha \oplus \ldots \oplus \mathbb{V}_\alpha}_{m_\alpha \text{ terms}}$$

We denote by $\mathbb{E}_\alpha$ this vector subspace of $\mathbb{V}$. We shall write

$$\mathbb{E}_\alpha = m_\alpha \mathbb{V}_\alpha = \bigoplus_{j=1}^{m_\alpha} \mathbb{V}_{\alpha,j}$$

where each $\mathbb{V}_{\alpha,j}$, $1 \leq j \leq m_\alpha$, is equal to $\mathbb{V}_\alpha$ and one particular $\mathbb{V}_{\alpha,j}$ corresponds to the j-th occurrence of vector space $\mathbb{V}_i$ in the direct sum of vector spaces presented above.

We thus have $\mathbb{V} = \bigoplus_{\alpha=1}^{N} \mathbb{E}_\alpha$. This decomposition is called the **isotypic decomposition** [19, 23].

Definition 2.11 (Isotypic components)

If ρ admits the decomposition

$$\rho = m_1 \rho^{(1)} \oplus m_2 \rho^{(2)} \oplus \ldots \oplus m_N \rho^{(N)}$$

then the non-negative integer m_α is the multiplicity of $\rho^{(\alpha)}$ in ρ, and $m_\alpha \rho^{(\alpha)}$ is the ***isotypic component*** *of type $\rho^{(\alpha)}$ of ρ.* ♣

Note *The decomposition into isotypic components is unique up to order.*

Example 2.14 Based on what we have just introduced, let's return to the notion of equivalent representation and consider the following representation $\left(\varrho_1 = \rho^{(2)} \otimes \rho^{(2)}, \mathbb{R}^2 \otimes \mathbb{R}^2\right)$. The corresponding matrix generators of D_3 are

$$[\varrho_1(\mathbf{r}_3)] = \begin{pmatrix} 1/4 & 3/4 & \sqrt{3}/4 & \sqrt{3}/4 \\ 3/4 & 1/4 & -\sqrt{3}/4 & -\sqrt{3}/4 \\ -\sqrt{3}/4 & \sqrt{3}/4 & 1/4 & -3/4 \\ -\sqrt{3}/4 & \sqrt{3}/4 & -3/4 & 1/4 \end{pmatrix}, \quad [\varrho_1(\boldsymbol{\pi})] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

This representation is reducible and **equivalent** to the following one $\left(\varrho_2 = \rho^{(e)} \oplus \rho^{(1)} \oplus \rho^{(2)}, A_1 \oplus A_2 \oplus E\right)$ for which

$$[\varrho_2(\mathbf{r}_3)] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1/2 & -\sqrt{3}/2 \\ 0 & 0 & \sqrt{3}/2 & -1/2 \end{pmatrix}, \quad [\varrho_2(\boldsymbol{\pi})] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

In this case the inversible change of basis is given by the orthogonal matrix

$$[T] = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Note *The intricate relation between an irreducible representation and the irreducible space support of this representation leads to the common abuse of notation where the same name is given to the irreducible representation and its associated irreducible space. As such, the irreducible space $\mathbb{E}_1$ can be noted sometimes A_1, for instance.*

2.4 Some irreducible representations for standard groups

In this section are presented tables of the irreducible representations for some standard finite groups. More examples can be find on the following sites : [8, 17], and their constructions are detailed in [29]

(a)

Z_2	e	$\mathbf{r}_2$ or $\boldsymbol{\pi}$
A	1	1
B	1	-1

(b)

D_2	e	$\mathbf{r}_2$	$\boldsymbol{\pi}$	$\mathbf{r}_2\boldsymbol{\pi}$
A_1	1	1	1	1
A_2	1	1	-1	-1
B_1	1	-1	1	-1
B_2	1	-1	-1	1

Table 2.1: Irreducible representations of groups (a) Z_2 and (b) D_2

Note *The abstract group $\mathbb{Z}_2$ may concern either a group with only one rotation or a group with only one reflection. Depending on the case, the irreducible representation names may be $\{A, B\}$ or $\{A_1, A_2\}$ depending on the geometrical interpretation of the elements.*

D_3	e	$\mathbf{r}_3$	$\mathbf{r}_3^2$	$\boldsymbol{\pi}$	$\mathbf{r}_3\boldsymbol{\pi}$	$\mathbf{r}_3^2\boldsymbol{\pi}$
A_1	1	1	1	1	1	1
A_2	1	1	1	-1	-1	-1
E	$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$	$\begin{pmatrix}c & -s\\s & c\end{pmatrix}$	$\begin{pmatrix}c & s\\-s & c\end{pmatrix}$	$\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$	$\begin{pmatrix}c & s\\s & -c\end{pmatrix}$	$\begin{pmatrix}c & -s\\-s & -c\end{pmatrix}$

Table 2.2: Irreducible representations of group D_3 where $c = \cos\frac{2\pi}{6} = \frac{1}{2}$ and $s = \sin\frac{2\pi}{6} = \frac{\sqrt{3}}{2}$

D_4	e	$\mathbf{r}_4$	$\mathbf{r}_2$	$\mathbf{r}_4^3$	$\boldsymbol{\pi}$	$\mathbf{r}_4\boldsymbol{\pi}$	$\mathbf{r}_2\boldsymbol{\pi}$	$\mathbf{r}_4^3\boldsymbol{\pi}$
A_1	1	1	1	1	1	1	1	1
A_2	1	1	1	1	-1	-1	-1	-1
B_1	1	-1	1	-1	1	-1	1	-1
B_2	1	-1	1	-1	-1	1	-1	1
E	$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$	$\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}$	$\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}$	$\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$	$\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$	$\begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}$	$\begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}$	$\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$

Table 2.3: Irreducible representations of group D_4

D_5	e	$\mathbf{r}_5$	$\mathbf{r}_5^2$	$\mathbf{r}_5^3$	$\mathbf{r}_5^4$	$\boldsymbol{\pi}$	$\mathbf{r}_5\boldsymbol{\pi}$	$\mathbf{r}_5^2\boldsymbol{\pi}$	$\mathbf{r}_5^3\boldsymbol{\pi}$	$\mathbf{r}_5^4\boldsymbol{\pi}$
A_1	1	1	1	1	1	1	1	1	1	1
A_2	1	1	1	1	1	-1	-1	-1	-1	-1
E_1	$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$	$\begin{pmatrix}c & -s\\s & c\end{pmatrix}$	$\begin{pmatrix}c^* & -s^*\\s^* & c^*\end{pmatrix}$	$\begin{pmatrix}c^* & s^*\\-s^* & c^*\end{pmatrix}$	$\begin{pmatrix}c & s\\-s & c\end{pmatrix}$	$\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$	$\begin{pmatrix}c & s\\s & -c\end{pmatrix}$	$\begin{pmatrix}c^* & s^*\\s^* & -c^*\end{pmatrix}$	$\begin{pmatrix}c^* & -s^*\\-s^* & -c^*\end{pmatrix}$	$\begin{pmatrix}c & -s\\-s & -c\end{pmatrix}$
E_2	$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$	$\begin{pmatrix}c^* & -s^*\\s^* & c^*\end{pmatrix}$	$\begin{pmatrix}c & s\\-s & c\end{pmatrix}$	$\begin{pmatrix}c & -s\\s & c\end{pmatrix}$	$\begin{pmatrix}c^* & s^*\\-s^* & c^*\end{pmatrix}$	$\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$	$\begin{pmatrix}c^* & s^*\\s^* & -c^*\end{pmatrix}$	$\begin{pmatrix}c & -s\\-s & -c\end{pmatrix}$	$\begin{pmatrix}c & s\\s & -c\end{pmatrix}$	$\begin{pmatrix}c^* & -s^*\\-s^* & -c^*\end{pmatrix}$

Table 2.4: Irreducible representations of group D_5 where $c = \cos\frac{2\pi}{5}$, $s = \sin\frac{2\pi}{5}$, $c^* = \cos\frac{4\pi}{5}$ and $s^* = \sin\frac{4\pi}{5}$

D_6	e	$\mathbf{r}_6$	$\mathbf{r}_3$	$\mathbf{r}_2$	$\mathbf{r}_3^2$	$\mathbf{r}_6^5$	$\boldsymbol{\pi}$	$\mathbf{r}_6\boldsymbol{\pi}$	$\mathbf{r}_3\boldsymbol{\pi}$	$\mathbf{r}_2\boldsymbol{\pi}$	$\mathbf{r}_3^2\boldsymbol{\pi}$	$\mathbf{r}_6^5\boldsymbol{\pi}$
A_1	1	1	1	1	1	1	1	1	1	1	1	1
A_2	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1
B_1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1
B_2	1	-1	1	-1	1	-1	-1	1	-1	1	-1	1
E_1	$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$	$\begin{pmatrix}c & -s\\s & c\end{pmatrix}$	$\begin{pmatrix}-c & -s\\s & -c\end{pmatrix}$	$\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}$	$\begin{pmatrix}-c & s\\-s & -c\end{pmatrix}$	$\begin{pmatrix}c & s\\-s & c\end{pmatrix}$	$\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$	$\begin{pmatrix}c & s\\s & -c\end{pmatrix}$	$\begin{pmatrix}-c & s\\s & c\end{pmatrix}$	$\begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}$	$\begin{pmatrix}-c & s\\s & c\end{pmatrix}$	$\begin{pmatrix}c & s\\s & -c\end{pmatrix}$
E_2	$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$	$\begin{pmatrix}c & -s\\s & c\end{pmatrix}$	$\begin{pmatrix}-c & -s\\s & -c\end{pmatrix}$	$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$	$\begin{pmatrix}c & -s\\s & c\end{pmatrix}$	$\begin{pmatrix}-c & -s\\s & -c\end{pmatrix}$	$\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$	$\begin{pmatrix}c & -s\\-s & -c\end{pmatrix}$	$\begin{pmatrix}-c & -s\\-s & c\end{pmatrix}$	$\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$	$\begin{pmatrix}c & -s\\-s & -c\end{pmatrix}$	$\begin{pmatrix}-c & -s\\-s & c\end{pmatrix}$

Table 2.5: Irreducible representations of group D_6

2.5 Restricted Representation

In group theory, restriction (also called subduction) forms a representation of a subgroup using a known representation of the whole group. Restriction is a fundamental construction in representation theory of groups. Often the restricted representation is simpler to understand. Rules for decomposing the restriction of an irreducible representation into irreducible representations of the subgroup are called **branching rules**, and have important applications in physics. For example, in case of explicit symmetry breaking, the symmetry group of the problem is reduced from the whole group to one of its subgroups.

The induced representation is a related operation that forms a representation of the whole group from a representation of a subgroup. The relation between restriction and induction is described by Frobenius reciprocity and the Mackey theorem. The induced representations and related theorems will not be presented in this book. Readers are directed to standard mathematical textbooks for more informations [16].

Definition 2.12 (Restriction)

For any group G*, its subgroup* H*, and a linear representation* ρ *of* G*, the restriction of* ρ *to* H*, denoted* $\rho|_H$ *is a representation of* H *on the same vector space by the same operators:*

$$\rho|_H(h) = \rho(h), \ \forall h \in \mathrm{H}$$

♣

It often happens that an irreducible representation of a group G restricts to a reducible representation of its subgroup H. This reducible representation can then be decomposed as a sum of irreducible representations of the parent group G. When this is done for all the irreducible representations of the parent group G, we have a subduction table that describes the branching rules.

Example 2.15 As an example, one could see how the irreducible representations of D_4 are restricted to D_2. The common elements between both groups are $\{e, \mathbf{r}_2, \boldsymbol{\pi}, \mathbf{r}_2\boldsymbol{\pi}\}$. While the four one-dimensional irreducible representations are common to both groups, D_4 presents a two-dimensional representation E that is reducible in D_2. Indeed, if one observed the restriction of that representation to the elements of D_2, it is obvious that two subspaces can be identified as this representation is always block-diagonal:

D_2	e	$\mathbf{r}_2$	$\boldsymbol{\pi}$	$\mathbf{r}_2\boldsymbol{\pi}$
A_1	1	1	1	1
A_2	1	1	-1	-1
B_1	1	-1	1	-1
B_2	1	-1	-1	1
E	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

One can then identify the following branching rules:

D_4	$\rightarrow$	D_2
A_1	$\rightarrow$	A_1
A_2	$\rightarrow$	A_2
B_1	$\rightarrow$	B_1
B_2	$\rightarrow$	B_2
E	$\rightarrow$	$B_1 \oplus B_2$

To make more general restrictions between finite groups, we need a powerful tool which comes from what is known as **character theory**.

3 Character theory

An essential problem in the study of linear representations and their application in mechanics is the determination of the isotypic decomposition. We need to know which irreducible representations appear in the decomposition and with what multiplicity. In the context of finite groups, character theory is the central tool for answering these questions.

3.1 Definitions

Consider $(\rho, \mathbb{V})$ a representation of the group G on the vector space $\mathbb{V}$. As we saw in the previous section, once a basis has been chosen for $\mathbb{V}$, the representation is described by a collection of matrices. An interesting invariant of matrices is their trace. This quantity is,

- invariant for similar matrices and therefore by conjugation;
- the sum of the eigenvalues of the matrix;
- allows the coefficients of the characteristic polynomial to be determined.

These properties, which motivated its use in representation theory, are at the heart of **character theory**. The traces of representation matrices are called the **characters of the representation**.

Definition 2.13 (Characters)

Let $\mathbb{V}$ be a finite-dimensional vector space and let $(\rho, \mathbb{V})$ be a representation of a group G on $\mathbb{V}$. The character of $(\rho, \mathbb{V})$ is the function $\chi_\rho : \mathrm{G} \to \mathbb{C}$ given by :

$$\chi_\rho(g) = \mathrm{tr}\,(\rho(g))$$

where tr is the trace. ♣

In practice, characters are calculated from the trace of the representation matrices expressed in a given base, but their values are independent of such a choice.

Note *When working with tensor representations, it is important to distinguish between the dimension of the vector space being tensored and the dimension of the vector space obtained by this operation. For example, the space $\mathbb{V} = \mathbb{R}^d \otimes \mathbb{R}^d$ is a vector space of dimension d^2 constructed as a tensor of order 2 on $\mathbb{R}^d$ of dimension d. So when we talk about the representation of $(\rho, \mathbb{V})$, one element of $\mathbb{V}$ is a vector of dimension d^2, and the matrix associated with $\rho(g)$ is a square matrix of dimension d. In character theory it is this trace that is calculated and unambiguously defined. This point is exemplified in the last example of subsection 2.1.*

Example 2.1 (Continued) Let's take the example Table 2.1 and consider the representation $(\rho^{(2)}, \mathbb{R}^2)$ of D_3. The following table give both the matrix representation and its associated characters for each element in the group.

D_3	e	$\mathbf{r}_3$	$\mathbf{r}_3^2$	$\boldsymbol{\pi}$	$\mathbf{r}_3\boldsymbol{\pi}$	$\mathbf{r}_3^2\boldsymbol{\pi}$
$\rho^{(2)}(g)$	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}-1/2 & -\sqrt{3}/2\\ \sqrt{3}/2 & -1/2\end{pmatrix}$	$\begin{pmatrix}-1/2 & \sqrt{3}/2\\ -\sqrt{3}/2 & -1/2\end{pmatrix}$	$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$	$\begin{pmatrix}-1/2 & -\sqrt{3}/2\\ -\sqrt{3}/2 & 1/2\end{pmatrix}$	$\begin{pmatrix}-1/2 & \sqrt{3}/2\\ \sqrt{3}/2 & 1/2\end{pmatrix}$
$\chi_{\rho^{(2)}}(g)$	2	-1	-1	0	0	0

On this result, the following facts can be observed:

1. $\chi_{\rho^{(2)}}(e) = 2$, which is the dimension of the representation ;
2. $\chi_{\rho^{(2)}}(\mathbf{r}_3) = \chi_{\rho^{(2)}}(\mathbf{r}_3^2) = -1$, the character takes the same values for the rotations;
3. $\chi_{\rho^{(2)}}(\boldsymbol{\pi}) = \chi_{\rho^{(2)}}(\mathbf{r}_3\boldsymbol{\pi}) = \chi_{\rho^{(2)}}(\mathbf{r}_3^2\boldsymbol{\pi}) = 0$, the character takes the same value for all the mirrors

As observed in the previous example, the following general properties of characters can be established:

Proposition 2.1

1. *the character of the identity element corresponds to the dimension of the representation: $\chi_\rho(e) = \dim \mathbb{V}$.*
2. *the character of the g^{-1} is the complex conjugate of the one of g:*
$$\chi(g^{-1}) = \overline{\chi(g)}$$
3. *the function χ_ρ is constant on each conjugacy class of G,*
$$\chi_\rho(hgh^{-1}) = \chi_\rho(g), \quad \forall g, h\mathrm{G}$$
*As such, characters are **class functions**;* ♠

Proof

1. For a representation $(\rho, \mathbb{V})$, $\chi_\rho(e) = \mathrm{tr}\mathrm{I}_{\mathbb{V}} = \dim \mathbb{V}$.
2. Since ρ is a group morphism, $\rho(g^{-1}) = \rho(g)^{-1}$. The representation being unitary implies $\rho(g)^{-1} = \overline{\rho}(g)^T$, hence $\chi(g^{-1}) =$

$\mathrm{tr}\overline{\rho}(g) = \overline{\mathrm{tr}\rho(g)} = \overline{\chi(g)}$.

3. The expression $\chi_\rho(hgh^{-1})$ can also be written $\chi_\rho(uv)$, putting $u = hg, v = h^{-1}$. Hence the result follows from the well known formula $\mathrm{tr}(uv) = \mathrm{tr}(vu)$.

□

When G is finite, the kernel of the character χ_ρ is the normal subgroup:

$$\ker \chi_\rho := \{g \in G \mid \chi_\rho(g) = \chi_{\rho(1)}\}$$

which is precisely the kernel of the representation ρ. However, the character is not a group homomorphism in general.

These former properties should be completed by the following ones concerning the characters of a direct sum and of a tensor product:

Proposition 2.2

direct sum: *the character of a direct sum of representations is the sum of the characters,*

$$\chi_{\rho^{(\alpha)} \oplus \rho^{(\beta)}} = \chi_{\rho^{(\alpha)}} + \chi_{\rho^{(\beta)}}$$

tensor product: *the character of a tensor product of representations is the product of the characters,*

$$\chi_{\rho^{(\alpha)} \otimes \rho^{(\beta)}} = \chi_{\rho^{(\alpha)}} \chi_{\rho^{(\beta)}}$$

Example 2.1 (Continued) Based on representations $\rho^{(1)}$ and $\rho^{(2)}$, the characters of $\rho^{(3)} = \rho^{(1)} \oplus \rho^{(2)}$ can be computed as $\chi_{\rho^{(3)}} = \chi_{\rho^{(1)}} + \chi_{\rho^{(2)}}$. Details are given in the table below:

D_3	e	$\mathbf{r}_3$	$\mathbf{r}_3^2$	$\boldsymbol{\pi}$	$\mathbf{r}_3\boldsymbol{\pi}$	$\mathbf{r}_3^2\boldsymbol{\pi}$
$\chi_{\rho^{(1)}}(g)$	1	1	1	1	1	1
$\chi_{\rho^{(2)}}(g)$	2	-1	-1	0	0	0
$\chi_{\rho^{(3)}}(g)$	3	0	0	1	1	1

Remark A character χ_ρ is called irreducible, or simple, if $(\rho, \mathbb{V})$ is an irreducible G-representation.

3.2 Orthogonality relations

We shall denote by $\mathcal{F}(\mathrm{G})$ the vector space of functions on the finite group G taking values in $\mathbb{C}$. For finite groups, it is a finite dimensional vector space of dimension the order of G. When this vector space is equipped with the scalar product defined below, we call the resulting space $L^2(\mathrm{G})$.

Definition 2.14

On $L^2(\mathrm{G})$, the scalar product is defined by

$$(f_1|f_2) = \frac{1}{|\mathrm{G}|} \sum_{g \in \mathrm{G}} \overline{f_1(g)} f_2(g).$$

with $\overline{a}$ stands for the complex conjugate of a.

Remark It can be observed that the considered scalar product is antilinear in the first argument and linear in the second. This is a convention and a scalar product where the antilinearity is in the second argument could be defined.

Note *It has to be stressed that we are using two different scalar product along this document. The first one is on the vector space $\mathbb{V}$ of the representation, and is denoted $<.,.>$, while the second is on the function space $L^2(\mathrm{G})$ and is denoted $(.|.)$.*

Since character functions are elements of $L^2(\mathrm{G})$, we can define the scalar product between character functions.

Definition 2.15

For representations $\rho^{(\alpha)}$ and $\rho^{(\beta)}$ of G

$$(\chi_{\rho^{(\alpha)}}|\chi_{\rho^{(\beta)}}) = \frac{1}{|\mathrm{G}|}\sum_{g\in \mathrm{G}} \chi_{\rho^{(\alpha)}}(g^{-1})\chi_{\rho^{(\beta)}}(g)$$

♣

This scalar product has the previous expression owing to the fact that, for character functions $\overline{\chi(g)} = \chi(g^{-1})$. In the case where characters are real, this simplifies into $\overline{\chi(g)} = \chi(g)$, otherwise stated as $\chi(g) = \chi(g^{-1})$

Remark The set of irreducible characters of a given group G into a field $\mathbb{k}$ form a basis of the $\mathbb{k}$-vector space of all class functions of G in $\mathbb{k}$.

Using Schur's Lemma the following central result, called **orthogonality relations** can be established. These relations state that the norm of an irreducible character is necessarily unitary and that the characters of inequivalent irreducible representations are orthogonal with respect to the scalar product defined above. We refer to classical textbooks for a detailed proof of this result [23]

Theorem 2.8 (Orthogonality Relations)

1. *If $\rho^{(\alpha)}$ and $\rho^{(\beta)}$ are inequivalent irreducible representations of* G*, then*

$$(\chi_{\rho^{(\alpha)}}|\chi_{\rho^{(\beta)}}) = 0.$$

2. *If ρ is an irreducible representations of* G*, then*

$$(\chi_{\rho}|\chi_{\rho}) = 1.$$

♡

For a group G whose irreducible characters are known, the orthogonality relations introduced above can be used to determine the isotypic decomposition of any representation of this same group. The multiplicity of an irreducible representation $\rho^{(\alpha)}$ in this decomposition is determined from the scalar product between the character of that irreducible representation and that of the reducible representation.

Theorem 2.9

Let ρ be any representation of G *and let χ_ρ be its character. Then*

$$\rho = \bigoplus_{\alpha=1}^{N} m_\alpha \rho^{(\alpha)},$$

where

$$m_\alpha = (\chi_{\rho^{(\alpha)}}|\chi_\rho) = \frac{1}{|\mathrm{G}|}\sum_{g\in \mathrm{G}} \chi_{\rho^{(\alpha)}}(g^{-1})\chi_\rho(g).$$

♡

Associated to this theorem we have the following and useful corollary which gives a necessary and sufficient condition for a representation to be irreducible.

Corollary 2.1

Let ρ be any representation of G *and let χ_ρ be its character, we have the formula*

$$(\chi_\rho|\chi_\rho) = \sum_{\alpha=1}^{N} m_\alpha^2$$

Irreducibility Criterion*: a representation ρ is irreducible if and only if*

$$(\chi_\rho|\chi_\rho) = 1$$

♡

Finally, this last theorem relates the dimension of the irreducible representations to the group order.

Theorem 2.10

Let denote $n_\alpha = \dim \mathbb{V}_\alpha$*, we have the following relation*

$$\sum_{\alpha=1}^{N} n_\alpha^2 = |\mathrm{G}|$$

3.3 Character tables for standard groups

In this last section are presented the character tables for some standard finite groups. Since characters are class functions, these tables are presented for classes represented by one of its representative element, instead of for each element. The number of elements in each class is denoted by $\#$.

(a)

Z_2	e	$\mathbf{r}_2$ or $\boldsymbol{\pi}$
A	1	1
B	1	-1

(b)

D_2	e	$\mathbf{r}_2$	$\boldsymbol{\pi}$	$\mathbf{r}_2\boldsymbol{\pi}$
A_1	1	1	1	1
A_2	1	1	-1	-1
B_1	1	-1	1	-1
B_2	1	-1	-1	1

Table 2.6: Character table for irreducible representations of groups (a) Z_2 and (b) D_2

Note *The attentive reader will note that the character table of the irreducible representations of groups* Z_2 *and* D_2 *are identical to the tables of the irreducible representations of these groups presented in Table 2.1. Indeed, characters are the trace of the irreducible representations and, as such, are equal to the one-dimensional irreducible representations.*

D_3	e	$\mathbf{r}_3$	$\boldsymbol{\pi}$
$\#$	1	2	3
A_1	1	1	1
A_2	1	1	-1
E	2	-1	0

Table 2.7: Character table for irreducible representations of group D_3

D_4	e	$\mathbf{r}_4$	$\mathbf{r}_2$	$\boldsymbol{\pi}$	$\mathbf{r}_4\boldsymbol{\pi}$
$\#$	1	2	1	2	2
A_1	1	1	1	1	1
A_2	1	1	1	-1	-1
B_1	1	-1	1	1	-1
B_2	1	-1	1	-1	1
E	2	0	-2	0	0

Table 2.8: Character tables for irreducible representations of group D_4

D_5	e	$\mathbf{r}_5$	$\mathbf{r}_5^2$	$\boldsymbol{\pi}$
$\#$	1	2	2	5
A_1	1	1	1	1
A_2	1	1	1	-1
E_1	2	$2\cos(2\pi/5)$	$2\cos(4\pi/5)$	0
E_2	2	$2\cos(4\pi/5)$	$2\cos(2\pi/5)$	0

Table 2.9: Character tables for irreducible representations of group D_5

D_6 $\#$	e 1	$\mathbf{r}_6$ 2	$\mathbf{r}_3$ 2	$\mathbf{r}_2$ 1	$\boldsymbol{\pi}$ 3	$\mathbf{r}_6\boldsymbol{\pi}$ 3
A_1	1	1	1	1	1	1
A_2	1	1	1	1	-1	-1
B_1	1	-1	1	-1	1	-1
B_2	1	-1	1	-1	-1	1
E_1	2	1	-1	-2	0	0
E_2	2	-1	-1	2	0	0

Table 2.10: Character tables for irreducible representations of group D_6

4 Extension to the orthogonal group in $\mathbb{R}^2$

In this section, we look at how to extend what has just been introduced to the case of the full **orthogonal group** in dimension 2, denoted by $\mathrm{O}(2)$. This group is defined as the set of linear isometries that preserve the scalar product of $\mathbb{R}^2$.

$$\forall g \in \mathrm{O}(2),\ < \rho(g)\underline{\mathrm{u}}, \rho(g)\underline{\mathrm{v}} > = < \underline{\mathrm{u}}, \underline{\mathrm{v}} >$$

in which $\rho(g)$ is the standard action on $\mathbb{R}^2$. In this situation ρ is often omitted.

The full orthogonal group in dimension 2 contains the set of rotations, which is a normal subgroup of index 2 denoted $\mathrm{SO}(2)$ and called the special orthogonal group, as well as the product of rotations by a mirror. Thus, we can equivalently see $\mathrm{O}(2)$:

- in terms of the decomposition into cosets $\mathrm{O}(2) = \mathrm{SO}(2) \cup \boldsymbol{\pi}\mathrm{SO}(2)$
- as the semi-direct product of Z_2 with $\mathrm{SO}(2)$, $\mathrm{O}(2) = \mathrm{Z}_2 \ltimes \mathrm{SO}(2)$

The finite diedral $\mathbb{D}_n$ and cyclic $\mathbb{Z}_n$ groups are finite subgroups of $\mathrm{O}(2)$.

$\mathrm{O}(2)$ is a continuous group which does not fit into the definitions established above. However, it is a compact group[1], i.e. a continuous group which behaves well. We refer you to the specialised references for the details, but, broadly speaking, everything happens more or less as in the finite case. We will see in the coming sections how the results we saw earlier extend to $\mathrm{O}(2)$.

4.1 Definition

The group $\mathrm{O}(2)$ is defined as a group of matrices as

$$\mathrm{O}(d) = \{\mathbf{g} \in \mathrm{GL}(2) | \mathbf{g}^T\mathbf{g} = \mathrm{I}_2\}$$

In particular, we have $\det \mathbf{g} = \pm 1$. The subset of matrices $\mathbf{g}$ such that $\det \mathbf{g} = 1$ is a subgroup of $\mathrm{O}(2)$, denoted by $\mathrm{SO}(2)$, which is the group of rotation, each rotation being represented by the matrix

$$[\mathbf{r}_\theta] := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{2.1}$$

The full orthogonal group is obtained from $\mathrm{SO}(2)$ by adding, as a generator, the matrix representing the reflection operation with respect to the horizontal axis

$$[\boldsymbol{\pi}] := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.2}$$

Besides, each element of $\mathrm{O}(2)$ can be written, either as $\mathbf{r}_\theta$ (if $\det g = 1$) or $\boldsymbol{\pi}\mathbf{r}_\theta$ (if $\det g = -1$), and we have the relations

$$\boldsymbol{\pi}^2 = e, \qquad \boldsymbol{\pi}\mathbf{r}_\theta = \mathbf{r}_{-\theta}\boldsymbol{\pi}$$

[1]The image of the unit ball by the group is bounded, which is obvious because $\det g \pm 1$ for $g \in \mathrm{O}(2)$. Note in passing that this is not true for $\mathrm{GL}(d)$ which is a non-compact group.

As demonstrated by the following lemma, geometrically elements of $\mathrm{O}(2)$ are of two type only: rotation (if $\det g = 1$) or reflection (if $\det g = -1$).

Lemma 2.1

The transformation $\mathbf{r}_\theta \boldsymbol{\pi}(\underline{n})$ is a reflection through the line of normal $\mathbf{r}_{\theta/2}\underline{n}$. ♡

Proof The plane of reflection $\boldsymbol{\pi}$ is of normal $\underline{e}_2$, since $\boldsymbol{\pi}(\underline{n}) = \underset{\sim}{\mathrm{I}} - 2\underline{n} \otimes \underline{n}$. We have the following relations

$$\mathbf{r}_\theta \boldsymbol{\pi}(\underline{n}) = \mathbf{r}_{\theta/2}\mathbf{r}_{\theta/2}\boldsymbol{\pi}(\underline{n}) = \mathbf{r}_{\theta/2}\boldsymbol{\pi}(\underline{n})\mathbf{r}_{-\theta/2} = \boldsymbol{\pi}(\mathbf{r}_{\theta/2}\underline{n})$$

□

Note *In $\mathbb{R}^3$ the inversion, $\mathbf{i} = -1$ is a transformation of determinant -1 distinct from a reflection. In $\mathbb{R}^2$, this not the case since it corresponds to the rotation $\mathbf{i} = \mathbf{r}_\pi$.*

4.2 $\mathrm{O}(2)$-representation

The situation considered in this course is the following one:

1. G is a compact group, $\mathrm{O}(2)$ here, but also $\mathrm{O}(3)$ when considering problems in $\mathbb{R}^3$
2. the representations are finite-dimensional, i.e. $\mathbb{V}$ is a finite-dimensional vector space

We then have the following definition

Definition 2.16 (Compact group)

Let G be a compact group. A finite dimensional representation of G is defined to be a vector space $\mathbb{V}$ and a group morphism $\rho : \mathrm{G} \to \mathrm{GL}(\mathbb{V})$ such that for every $v \in \mathbb{V}$,

$$g \in \mathrm{G} \mapsto \rho(g)v \in \mathbb{V}$$

is a continuous mapping. ♣

This means that, as soon as a basis is considered for $\mathbb{V}$, the representations are always described by square matrices. The difference is that this set is now infinite and described by matrix-valued functions depending on the parameters of the group.

Under these assumptions we have the following theorems which indicate that the situation is the same as in the case of finite groups. For more details, we refer you to the associated mathematics books.

Theorem 2.11

Every finite-dimensional representation of a compact group is completely reducible. ♡

Theorem 2.12

Every irreducible representation of a compact group is finite dimensional. ♡

4.3 Irreducible representations

Real irreducible representations of 2D orthogonal groups are well-known (see for instance [19]). We have the following results:

SO(2) Each real irreducible representation of $\mathrm{SO}(2)$ is either equivalent to the **trivial representation** on $\mathbb{R}$, denoted by $\rho^{(0)}$ and defined by

$$\rho^{(0)}(g)\lambda = \lambda, \text{ for all } g \in \mathrm{SO}(2) \text{ and all } \lambda \in \mathbb{R}$$

or to the two-dimensional representation on $\mathbb{R}^2$ given by

$$\mathbf{r}_\theta \in \mathrm{SO}(2) \mapsto \rho^{(n)}(\mathbf{r}_\theta) = \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix} \in \mathrm{GL}(\mathbb{R}^2),$$

and indexed by the integer $n \geq 1$.

O(2) Each real irreducible representations of $\mathrm{O}(2)$ is either equivalent to the **trivial representation** on $\mathbb{R}$, denoted by $\rho^{(0)}$, the **sign representation** on $\mathbb{R}$, denoted by $\rho^{(-1)}$ and defined by

$$\rho^{(-1)}(g)\xi = (\det g)\xi,$$

for all $g \in \mathrm{O}(2)$ and all $\xi \in \mathbb{R}$ (ξ is sometimes called a pseudo-scalar), or to the following representations on $\mathbb{R}^2$ given by

$$\rho^{(n)}(\mathbf{r}_\theta) = \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix}, \quad \rho^{(n)}(\boldsymbol{\pi}\mathbf{r}_\theta) = \begin{pmatrix} \cos n\theta & \sin n\theta \\ \sin n\theta & -\cos n\theta \end{pmatrix},$$

and indexed by the integer $n \geq 1$.

Note *The notations for these irreducible representations are different to the ones chosen for the finite groups. The trivial representation is denoted $\rho^{(0)}$ and not A_1. Similarly, the sign representation is denoted $\rho^{(-1)}$ and not A_2. This choice has been made to correspond to standard notations of harmonic spaces in $\mathbb{R}^2$ [3, 11].*

The characters of the irreducible representations $(\rho^{(n)}, \mathbb{K}^n)$ $(n = -1, 0, \dots)$ of $\mathrm{O}(2)$ have the following explicit expressions

$$\chi_{\rho^{(n)}}(\mathbf{r}_\theta) = 2\cos(n\theta), \quad \chi_{\rho^{(n)}}(\boldsymbol{\pi}\mathbf{r}_\theta) = 0, \quad (n \geq 1) \tag{2.3}$$

and

$$\chi_{\rho^{(0)}} = 1, \quad \chi_{\rho^{(-1)}}(\mathbf{r}_\theta) = 1, \quad \chi_{\rho^{(-1)}}(\boldsymbol{\pi}\mathbf{r}_\theta) = -1. \tag{2.4}$$

4.4 Model spaces

There are two models, useful in practice, for 2-dimensional irreducible representations of the orthogonal groups:

1. The spaces $\mathcal{H}^n$ of **homogeneous harmonic polynomials** (polynomials with vanishing Laplacian) in two variables x, y of degree $n \geq 1$,
2. The spaces $\mathbb{K}^n$ of nth-order **harmonic tensors** (totally symmetric tensors with vanishing traces).

And, to complete these alternative models for $n = 0$ and $n = -1$, we set

$$\mathbb{K}^0 = \mathcal{H}^0 = \mathbb{R}, \quad \text{with the trivial representation,}$$

and

$$\mathbb{K}^{-1} = \mathcal{H}^{-1} = \mathbb{R}, \quad \text{with the sign representation.}$$

Definition 2.17 (Harmonic tensor space)

Let $\mathbb{K}^n$ be the space of nth-order harmonic tensors in 2D, its elements are:

1. *n-th order tensors;*
2. *symmetric with respect to the permutation of all the indices;*
3. *traceless.*

Proposition 2.3

In $\mathbb{R}^2$, we have

$$\dim \mathbb{K}^n = \begin{cases} 2, & n \geq 1; \\ 1, & n = \{0, -1\}. \end{cases}$$

Any linear representation $(\rho, \mathbb{V})$ of $\mathrm{SO}(2)$ or $\mathrm{O}(2)$ can be decomposed into a direct sum of irreducible representations.

This is known as the **harmonic decomposition** of $\mathbb{V}$ and means that

$$\mathbb{V} \simeq \mathbb{K}^{n_1} \oplus \cdots \oplus \mathbb{K}^{n_p}, \tag{2.5}$$

where $n_i \in \{-1, 0, 1, 2, \dots\}$ and where multiplicities are allowed.

Grouping together identical harmonic spaces we obtain the **isotypic decomposition**, also known as the **harmonic structure** in the present context,

$$\mathbb{V} \simeq m_{-1}\mathbb{K}^{-1} \oplus m_0\mathbb{K}^0 \oplus \cdots \oplus m_n\mathbb{K}^{k_n}, \quad m_i\mathbb{K}^{k_i} := \bigoplus_{l=1}^{m_i} \mathbb{K}^{k_i}. \tag{2.6}$$

4.5 Harmonic structure

Given a representation ρ on a space $\mathbb{V}$, the question is how to determine its $\mathrm{O}(2)$ isotypic structure. First, we will extend the approach developed in the previous section for finite groups to this new situation. This means extending the theory of characters to $\mathrm{O}(2)$ and obtaining multiplicities by using an adapted scalar product. As we shall see, modulo the introduction of an integral instead of a finite sum, everything happens as in finite dimension. However, calculating the character of the representation, especially in the case of high-order symmetrized tensor representations, will quickly become complicated. In a second time, an approach based on so-called Clebsch-Gordan formulae will be introduced. These formulae, which we will prove from the characters, will make it trivial to establish the decompositions of tensorial representations.

4.5.1 Character theory

We denote by $\mathcal{F}(\mathrm{G})$ the vector space of complex-valued functions on G. For continuous compact groups, it is an infinite dimensional vector space. When this vector space is equipped with the inner product defined below, we call the resulting space $L^2(\mathrm{G})$.

Definition 2.18 (Inner product)

On $L^2(\mathrm{G})$, the scalar product is defined by

$$(f_1|f_2) := \int_{\mathrm{G}} f_1(g)f_2(g)\, d\mu(g). \tag{2.7}$$

where $d\mu(g)$ is the Haar measure. ♣

In the case $\mathrm{G} = \mathrm{O}(2)$ we have the following fundamental result:

Theorem 2.13

On $L^2(\mathrm{O}(2))$, the inner product is explicitly given by

$$(f_1|f_2) := \frac{1}{4\pi}\int_0^{2\pi} f_1(\mathbf{r}_\theta)f_2(\mathbf{r}_\theta)\, d\theta + \frac{1}{4\pi}\int_0^{2\pi} f_1(\boldsymbol{\pi}\mathbf{r}_\theta)f_2(\boldsymbol{\pi}\mathbf{r}_\theta)\, d\theta.$$

♡

If $(\rho, \mathbb{V})$ is a finite-dimensional representation of $\mathrm{O}(2)$, we define the character χ_ρ of ρ by

$$\forall g \in \mathrm{O}(2),\ \chi_\rho(g) = \mathrm{tr}(\rho(g)).$$

The complex-valued function thus defined on $\mathrm{O}(2)$ is continuous. It is a class function that depends only on the equivalence class of the representation ρ. As for finite groups, we have the following orthogonality relations:

Theorem 2.14 (Orthogonality Relations)

Let ρ_1 and ρ_2 be irreducible representations of $\mathrm{O}(2)$. *Then*

$$(\chi_{\rho_1}|\chi_{\rho_2}) = \begin{cases} 0 \text{ if } \rho_1 \neq \rho_2, \\ 1 \text{ if } \rho_1 \simeq \rho_2, \end{cases}$$

♡

and its corrolaries:

Lemma 2.2

A $\mathrm{O}(2)$*-representation* ρ *is irreducible if and only if* $(\chi_\rho|\chi_\rho) = 1$

Consider a representation $(\rho, \mathbb{V})$ and denote by χ_ρ its character, the multiplicity m_i of the irreducible representation $\mathbb{K}^{k_i}$ in the isotypic structure Equation 2.6 of $\mathbb{V}$ is given by:

$$m_i = \left(\chi_{\rho^{(i)}|\chi_\rho}\right).$$

Example 2.14 Consider $(\rho^{(2)}, \mathbb{R}^2)$ the standard action of $\mathrm{O}(2)$ on $\mathbb{R}^2$. For the sake of simplicity, the classical abuse of notation will be used:

$$\forall g \in \mathrm{O}(2),\ \rho^{(2)}_{ij}(g) = g_{ij}$$

The characters of this representation are the following:

$$\chi_{\rho^{(2)}}(\mathbf{r}_\theta) = 2\cos\theta, \quad \chi_{\rho^{(2)}}(\mathbf{r}_\theta\boldsymbol{\pi}) = 0$$

The norm of the non-zero character can be computed.

$$(\chi_{\rho^{(2)}}|\chi_{\rho^{(2)}}) = \frac{1}{\pi}\int_0^{2\pi} \cos^2\theta\, d\theta = 1$$

Hence, and obviously the representation is irreducible.

Example 2.15 Consider $(\rho, \mathbb{R}^2 \otimes \mathbb{R}^2)$ the tensor action of $\mathrm{O}(2)$ on $\mathbb{R}^2 \otimes \mathbb{R}^2$. This representation can be constructed by tensor product as presented in Definition 2.7:

$$\forall g \in \mathrm{O}(2),\ \rho_{ijkl}(g) = g_{ik}g_{jl}$$

The trace of ρ has the following expression

$$\mathrm{tr}\rho(g) = \rho_{ijij}(g) = (\mathrm{tr} g)^2$$

and

$$\chi_\rho(\mathbf{r}\theta) = 4\cos^2\theta = 2(1+\cos 2\theta), \quad \chi_\rho(\mathbf{r}_\theta\boldsymbol{\pi}) = 0$$

The norm of the non-zero character can be computed.

$$(\chi_\rho|\chi_\rho) = \frac{4}{\pi}\int_0^{2\pi} \cos^4\theta\, d\theta = 3$$

Hence the representation is reducible. Its various projections on irreducible characters are listed as follows.

	$\chi_{\rho^{(-1)}}$	$\chi_{\rho^{(0)}}$	$\chi_{\rho^{(1)}}$	$\chi_{\rho^{(2)}}$	$\chi_{\rho^{(n)}}$
χ_ρ	1	1	0	1	0

Hence the representation decomposes as follows

$$\rho^{(1)} \otimes \rho^{(1)} = \rho^{(-1)} \oplus \rho^{(0)} \oplus \rho^{(2)}$$

and we obtain the corresponding harmonic structure

$$\mathbb{R}^2 \otimes \mathbb{R}^2 = \mathbb{K}^{-1} \oplus \mathbb{K}^0 \oplus \mathbb{K}^2.$$

With this last result we retrieve the well-known fact that a second order tensor decomposes as a deviatoric tensor ($\mathbb{K}^2$), a hydrostatic part ($\mathbb{K}^0$) and an anti-symmetric part which in $\mathbb{R}^2$ is a pseudo scalar ($\mathbb{K}^{-1}$). The decomposition is illustrated below

$$\underbrace{\begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}}_{\in \mathbb{R}^2\otimes\mathbb{R}^2} = \underbrace{\frac{1}{2}\begin{pmatrix} t_{11}-t_{22} & t_{12}+t_{21} \\ t_{12}+t_{21} & -t_{11}+t_{22} \end{pmatrix}}_{\in \mathbb{K}^2} + \underbrace{\frac{1}{2}(t_{11}+t_{22})\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\in \mathbb{K}^0} + \underbrace{\frac{1}{2}(t_{12}-t_{21})\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{\in \mathbb{K}^{-1}}$$

In terms of transformation, the elements of $\mathbb{R}^2 \otimes \mathbb{R}^2$, considered as vectors in $\mathbb{R}^4$, are transformed as follows:

$$\begin{pmatrix} t_1^\star \\ t_2^\star \\ t_3^\star \\ t_4^\star \end{pmatrix} = \begin{pmatrix} \cos 2\theta & -\sin 2\theta & 0 & 0 \\ \sin 2\theta & \cos 2\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \det(g) \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix}, \text{ with } \begin{cases} t_1 = \frac{1}{2}(t_{11} - t_{22}) \\ t_2 = \frac{1}{2}(t_{12} + t_{21}) \\ t_3 = \frac{1}{2}(t_{11} + t_{22}) \\ t_4 = \frac{1}{2}(t_{12} - t_{21}) \end{cases}$$

The following example is the symmetrised version of the previous one. It shows that in this case the anti-symmetrical part disappears, as it should.

Example 2.16 Consider $(\rho, S^2(\mathbb{R}^2))$ the tensor action of $\mathrm{O}(2)$ on $\mathbb{R}^2 \otimes^s \mathbb{R}^2$. Once again this action can be constructed by tensor product and symmetrisation:

$$\forall g \in \mathrm{O}(2),\ \rho_{ijkl}(g) = \frac{1}{2}\left(g_{ik}g_{jl} + g_{il}g_{jk}\right)$$

The trace of ρ has the following expression

$$\mathrm{tr}\rho(g) = \rho_{ijij} = \frac{1}{2}\left((\mathrm{tr}g)^2 + \mathrm{tr}(g^2)\right)$$

and

$$\chi_{\rho(\mathbf{r}\theta)} = 1 + 2\cos 2\theta, \quad \chi_{\rho(\mathbf{r}_\theta \boldsymbol{\pi})} = 1$$

The norm of the character can be computed:

$$(\chi_\rho|\chi_\rho) = \frac{1}{4\pi}\int_0^{2\pi} (1 + 2\cos 2\theta)^2\, d\theta + \frac{1}{4\pi}\int_0^{2\pi} d\theta = 2$$

Hence the representation is reducible. Its various projections on irreducible characters are listed as follows

	$\chi_{\rho^{(-1)}}$	$\chi_{\rho^{(0)}}$	$\chi_{\rho^{(1)}}$	$\chi_{\rho^{(2)}}$	$\chi_{\rho^{(n)}}$
χ_ρ	0	1	0	1	0

Hence the representation decomposes as follows

$$\rho^{(1)} \otimes^s \rho^{(1)} = \rho^{(0)} \oplus \rho^{(2)}$$

and we obtain the following harmonic structure

$$\mathbb{R}^2 \otimes \mathbb{R}^2 = \mathbb{K}^0 \oplus \mathbb{K}^2.$$

To conclude this subsection let consider the more involved example of the elasticity tensor in $\mathbb{R}^2$.

Example 2.17 Consider $(\rho, \mathbb{E}\mathrm{la})$ the tensor action of $\mathrm{O}(2)$ on $S^2(S^2(\mathbb{R}^2))$:

$$(\rho(\mathbf{g}))_{ijklmnop} = \frac{1}{8}\left((g_{im}g_{jn} + g_{in}g_{jm})(g_{ko}g_{lp} + g_{kp}g_{lo}) + (g_{io}g_{jp} + g_{ip}g_{jo})(g_{km}g_{ln} + g_{kn}g_{lm})\right)$$

The trace of ρ has the following expression

$$\mathrm{tr}\rho(g) = \frac{1}{8}\left[(\mathrm{tr}g)^4 + 2(\mathrm{tr}g)^2\mathrm{tr}g^2 + 2trg^4 + 3(\mathrm{tr}g^2)^2\right]$$

Its various projection on irreducible characters are listed as follows

	$\chi_{\rho^{(-1)}}$	$\chi_{\rho^{(0)}}$	$\chi_{\rho^{(1)}}$	$\chi_{\rho^{(2)}}$	$\chi_{\rho^{(3)}}$	$\chi_{\rho^{(4)}}$	$\chi_{\rho^{(n)}}$
χ_ρ	0	2	0	1	0	1	0

Hence the representation decomposes as follows

$$\rho = 2\rho^{(0)} \oplus \rho^{(2)} \oplus \rho^{(4)}$$

and we obtain the following harmonic structure

$$\mathbb{E}\mathrm{la} \simeq 2\mathbb{K}^0 \oplus \mathbb{K}^2 \oplus \mathbb{K}^4$$

In terms of transformations, the elements of $\mathbb{E}\mathrm{la}$, considered as vectors in $\mathbb{R}^6$, are transformed as follows:

$$\begin{pmatrix} t_1^\star \\ t_2^\star \\ t_3^\star \\ t_4^\star \\ t_5^\star \\ t_6^\star \end{pmatrix} = \begin{pmatrix} \cos 4\theta & -\sin 4\theta & 0 & 0 & 0 & 0 \\ \sin 4\theta & \cos 4\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos 2\theta & -\sin 2\theta & 0 & 0 \\ 0 & 0 & \sin 2\theta & \cos 2\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \\ t_6 \end{pmatrix}$$

As can be observed on the previous example, the tensorial expression of the representation is rather complicated. This is due to two points:

1. the tensorial order of $\rho(\mathbf{g})$ is twice the order of the representation;
2. the index symmetries to consider imply complicated symmetrised expressions.

Consequently, the study of the action on tensors of order 6 or higher with index symmetries will quickly prove complicated, if not impossible. Therefore, in order to obtain isotypic high-order tensors, another method needs to be developed. This is the subject of the following section.

4.5.2 Clebsch-Gordan

It is a classical result of the theory of group representations that the product of two irreducible spaces is, in general, non-irreducible [23]. But the resulting space can in turn be decomposed into irreducible spaces. The formula expressing how the tensor product of irreducible spaces decomposes into irreducible spaces is known as the Clebsch-Gordan formula. For a general group, these formulas can be difficult to express, but for $\mathrm{O}(2)$ they are quite simple. For basic determination of harmonic structure, we use the following result:

Lemma 2.3 (Clebsch-Gordan formula)

The tensor product of two $\mathrm{O}(2)$-irreducible spaces is reducible and decomposes according to:

$\otimes$	$\mathbb{K}^n$	$\mathbb{K}^0$	$\mathbb{K}^{-1}$
$\mathbb{K}^m$	$\begin{cases} \mathbb{K}^{m+n} \oplus \mathbb{K}^{\lvert m-n\rvert}, & m \neq n \\ \mathbb{K}^{2n} \oplus \mathbb{K}^0 \oplus \mathbb{K}^{-1}, & m = n \end{cases}$	$\mathbb{K}^m$	$\mathbb{K}^m$
$\mathbb{K}^0$	$\mathbb{K}^n$	$\mathbb{K}^0$	$\mathbb{K}^{-1}$
$\mathbb{K}^{-1}$	$\mathbb{K}^n$	$\mathbb{K}^{-1}$	$\mathbb{K}^0$

♡

Proof Let us compute the characters associated to the various situations indicated in the table:

	$\mathbf{r}_\theta$	$\mathbf{r}_\theta\boldsymbol{\pi}$
$\chi_{\rho^{(p)}} \otimes \chi_{\rho^{(q)}}$	$4\cos p\theta \cos q\theta$	0
$\chi_{\rho^{(p)}} \otimes \chi_{\rho^{(0)}}$	$2\cos p\theta$	0
$\chi_{\rho^{(p)}} \otimes \chi_{\rho^{(-1)}}$	$2\cos p\theta$	0
$\chi_{\rho^{(0)}} \otimes \chi_{\rho^{(0)}}$	1	1
$\chi_{\rho^{(0)}} \otimes \chi_{\rho^{(-1)}}$	1	-1
$\chi_{\rho^{(-1)}} \otimes \chi_{\rho^{(-1)}}$	1	1

The only product that should be detailed concerns the representation $(\rho^{(p)} \otimes \rho^{(q)}, \mathbb{K}^p \otimes \mathbb{K}^q)$. The character of the representation is

$$\begin{aligned} \chi_{\rho^{(p)}\otimes\rho^{(q)}} &= \chi_{\rho^{(p)}}\chi_{\rho^{(q)}} = 4\cos p\theta \cos q\theta \\ &= 2\cos(p+q)\theta + 2\cos(p-q)\theta = \chi_{\rho^{(p+q)}} + \chi_{\rho^{(p-q)}} \end{aligned}$$

Hence

$$(\rho^{(p)} \otimes \rho^{(q)}, \mathbb{K}^p \otimes \mathbb{K}^q) = (\rho^{(p+q)} \oplus \rho^{(p-q)}, \mathbb{K}^{p+q} \oplus \mathbb{K}^{p-q})$$

□

Example 2.18 Using this rule the following results are directly obtained:

- for $\mathbb{R}^2 \otimes \mathbb{R}^2$

$$\mathbb{T}_{ij} \simeq \mathbb{K}^{-1} \oplus \mathbb{K}^0 \oplus \mathbb{K}^2,$$

  the dimension is 4 as it should be;
- for $(\mathbb{R}^2)^{\otimes^3}$

$$\mathbb{T}_{ijkk} \simeq (\mathbb{K}^1 \otimes \mathbb{K}^1) \otimes \mathbb{K}^1 = (\mathbb{K}^{-1} \oplus \mathbb{K}^0 \oplus \mathbb{K}^2) \otimes \mathbb{K}^1 = 3\mathbb{K}^1 \oplus \mathbb{K}^3$$

  the dimension is 8 as it should be;
- for $(\mathbb{R}^2)^{\otimes^4}$

$$\begin{aligned} \mathbb{T}_{ijkl} &\simeq (\mathbb{K}^1 \otimes \mathbb{K}^1 \otimes \mathbb{K}^1) \otimes \mathbb{K}^1) \\ &= (3\mathbb{K}^1 \oplus \mathbb{K}^3) \otimes \mathbb{K}^1 \\ &= 3\mathbb{K}^{-1} \oplus 3\mathbb{K}^0 \oplus 4\mathbb{K}^2 \oplus \mathbb{K}^4, \end{aligned}$$

  the dimension is 16 as it should be.

In the case the spaces are identical, the tensor product can be decomposed into S^2 and Λ^2. This represents, respectively, the symmetrised product and the anti-symmetrised product[2]:

$$\forall n \geq 1, \mathbb{K}^n \otimes \mathbb{K}^n = S^2(\mathbb{K}^n) \oplus \Lambda^2(\mathbb{K}^n)$$

Therefore, Lemma 2.3 is completed by the following lemma:

Lemma 2.4 (Clebsch-Gordan formula)

For all $n \geq 1$, we have the following isotropic decompositions, in which meaningless products are indicated by $\times$:

S^2	$\mathbb{K}^n$	$\mathbb{K}^0$	$\mathbb{K}^{-1}$
$\mathbb{K}^n$	$\mathbb{K}^{2n} \oplus \mathbb{K}^0$	$\times$	$\times$
$\mathbb{K}^0$	$\times$	$\mathbb{K}^0$	$\times$
$\mathbb{K}^{-1}$	$\times$	$\times$	$\mathbb{K}^0$

Λ^2	$\mathbb{K}^n$	$\mathbb{K}^0$	$\mathbb{K}^{-1}$
$\mathbb{K}^n$	$\mathbb{K}^{-1}$	$\times$	$\times$
$\mathbb{K}^0$	$\times$	0	$\times$
$\mathbb{K}^{-1}$	$\times$	$\times$	0

♡

Example 2.19 Consider the space $\mathbb{V} = S^2(S^2(\mathbb{R}^2) \otimes \mathbb{R}^2)$, it is a space of sixth-order tensors having the following index symmetries:

$$T_{\underline{(ij)k(lm)n}}$$

The harmonic structure can be worked out easily using Clebsch-Gordan formula.

- First step:

$$S^2(\mathbb{R}^2) = \mathbb{K}^0 \oplus \mathbb{K}^2$$

- Second step:

$$S^2(\mathbb{R}^2) \otimes \mathbb{R}^2 = (\mathbb{K}^0 \oplus \mathbb{K}^2) \otimes \mathbb{K}^1) = 2\mathbb{K}^1 \oplus \mathbb{K}^3$$

[2]Let $\mathbb{V}$ a vector space of dimension d, and $\underline{\mathrm{v_i}}$ the basis of $\mathbb{V}$, thus the basis of $\mathbb{V} \otimes \mathbb{V}$ is defined by: $\mathcal{B}(\mathbb{V} \otimes \mathbb{V}) = \underline{\mathrm{v_i}} \otimes \underline{\mathrm{v_j}}$, we have:

$$\mathcal{B}(S^2(\mathbb{V})) = \frac{1}{2}(\underline{\mathrm{v_i}} \otimes \underline{\mathrm{v_j}} + \underline{\mathrm{v_j}} \otimes \underline{\mathrm{v_i}}) \quad \mathcal{B}(\Lambda^2(\mathbb{V})) = \frac{1}{2}(\underline{\mathrm{v_i}} \otimes \underline{\mathrm{v_j}} - \underline{\mathrm{v_j}} \otimes \underline{\mathrm{v_i}})$$

- Third step:

$$S^2(S^2(\mathbb{R}^2) \otimes \mathbb{R}^2) = S^2(2\mathbb{K}^1 \oplus \mathbb{K}^3) = \mathbb{K}^{-1} \oplus 4\mathbb{K}^0 \oplus 5\mathbb{K}^2 \oplus 2\mathbb{K}^4 \oplus \mathbb{K}^6$$

Computing the same result using brut character theory would have been far more complicated.

Chapter 3 Applications to tensors in mechanics

Content

❏ *Strain-gradient elasticity law*

The aim of this short chapter is to apply the theoretical tools introduced in the previous chapters to the determination of the number of independent constants for different symmetry classes of Mindlin strain gradient elasticity. For the purposes of this course, we will restrict ourselves to the $\mathbb{R}^2$ framework, but the approach could easily be extended to $\mathbb{R}^3$.

The aim of this chapter is to compare two different approaches to this work. It should be noted that the determination of the number of invariant coefficients for a given subgroup class does not constitute a proof for determining the classes of symmetries of a constitutive law. We will assume here that the symmetry classes are known and we will calculate the dimension of the associated invariant space in each of these situations.

In the literature, there are traditionally two ways of proceeding: the top-down method and the bottom-up method:

top-down The **top-down** approach starts by establishing the isotypic decomposition with respect to $\mathrm{O}(2)$, before studying its restriction with respect to its subgroups, which are mostly finite. This approach is based on the harmonic decomposition.

bottom-up The **bottom-up** approach, based on the character theory for finite groups, and compute isotypic decomposition for finite groups from irreducible characters. This approach has its origin in the methods developed in crystallography to study the properties of molecules and lattices.

The first method aims to describe the infinitesimal transformation properties of the elements of a vector space. From this global description, particular results are the obtained by restriction. The use of this approach is classical in condensed matter physics [22] but is less classical in solid mechanics. In the mechanics literature, this approach is mainly used in theoretical documents to obtain classification theorems, to select one modelling strategy over another, to calculate the integrity basis, etc. Even if introduced in mechanics by Backus [6], the reference papers concerning this approach are those of Forte and Vianello [15]

In contrast, the second approach is mainly concerned with the modelling of a particular situation in which the symmetry group of the phenomenon has been identified and for which obtaining general results is not the aim. In the mechanical literature, this approach is manly used in stability analysis for understanding the behaviour of a particular structure or of an architectured material that possess geometrical symmetries. As this approach is rooted in crystallography, it is popular with engineers because of their training.

This chapter will be organised as follows. In a first section the equations of the investigated constitutive law will be detailed, together with results concerning the symmetry classes of the associated constitutive tensors. In the next section the number of invariant parameters for each symmetry classes is determined, first by the crystallographic bottom up approach, second by the harmonic top bottom one.

1 Strain-gradient elasticity law

We introduce in this section the constitutive law of a linear strain-gradient elastic material [26, 27].

State tensors describe point-wisely the different physical fields (primal and dual) of the model. A linear constitutive law can be viewed as a linear map between the state tensors that characterise a chosen physical model. A linear constitutive law is defined by a set of **constitutive tensors** which describe the influence of the matter on these state tensor fields, more precisely they describe how primal and dual fields are connected by the matter.

In the case of **classical elasticity**, the state tensors are $\underset{\sim}{\sigma}$, $\underset{\sim}{\varepsilon}$ and characterise the local state of stress and of strain, respectively[1]. These **state tensors** belong to the same space $\mathbb{S}^2$. The linearity of the model implies the use of a fourth-order tensor $\underset{\approx}{\mathrm{C}}$ as a constitutive tensor, this tensor can be viewed as an element of $\mathcal{L}^s(\mathbb{S}^2, \mathbb{S}^2)$. In summary, for classical elasticity:

- **State tensors**: $\underset{\sim}{\sigma}$, $\underset{\sim}{\varepsilon}$;
- **Constitutive tensor**: $\underset{\approx}{\mathrm{C}}$.

The linear strain-gradient elasticity model [26, 27] is obtained by extending the set of state tensors by including the strain-gradient tensor $\underset{\simeq}{\eta} := \underset{\sim}{\varepsilon} \otimes \underline{\nabla}$ and its dual quantity, the hyperstress tensor $\underset{\simeq}{\tau}$. Those tensors are elements of $\mathbb{S}^2 \otimes \mathbb{R}^2$. The constitutive equations of the model define the stress tensor $\underset{\sim}{\sigma}$ and the hyperstress tensor $\underset{\simeq}{\tau}$ as linear functions of the strain tensor $\underset{\sim}{\varepsilon}$ and the strain-gradient tensor $\underset{\simeq}{\eta}$. This coupled constitutive law requires tensors belonging to the following spaces

$$\underset{\approx}{\mathrm{C}} \in \mathbb{E}\mathrm{la}_4 \simeq \mathcal{L}^s(\mathbb{S}^2, \mathbb{S}^2), \quad \underset{\cong}{\mathrm{M}} \in \mathbb{E}\mathrm{la}_5 \simeq \mathcal{L}(\mathbb{S}^2 \otimes \mathbb{R}^2, \mathbb{S}^2), \quad \underset{\approxeq}{\mathrm{A}} \in \mathbb{E}\mathrm{la}_6 \simeq \mathcal{L}^s(\mathbb{S}^2 \otimes \mathbb{R}^2, \mathbb{S}^2 \otimes \mathbb{R}^2).$$

In this model we have:

- **State tensors**: $\underset{\sim}{\sigma}$, $\underset{\sim}{\varepsilon}$, $\underset{\simeq}{\tau}$, $\underset{\simeq}{\eta}$;
- **Constitutive tensors**: $\underset{\approx}{\mathrm{C}}$, $\underset{\cong}{\mathrm{M}}$ and $\underset{\approxeq}{\mathrm{A}}$.

To be more specific, the constitutive equations read:

$$\begin{cases} \underset{\sim}{\sigma} &= \underset{\approx}{\mathrm{C}} : \underset{\sim}{\varepsilon} + \underset{\cong}{\mathrm{M}} \vdots \underset{\simeq}{\eta} \\ \underset{\simeq}{\tau} &= \underset{\cong}{\mathrm{M}}^\top : \underset{\sim}{\varepsilon} + \underset{\approxeq}{\mathrm{A}} \vdots \underset{\simeq}{\eta} \end{cases} \tag{3.1}$$

where

- $\underset{\approx}{\mathrm{C}} \in \mathbb{E}\mathrm{la}_4 := \{\underset{\approx}{\mathrm{T}} \in \overset{4}{\otimes}\mathbb{R}^2 \mid \underset{\approx}{\mathrm{T}} \in \mathbb{T}_{\underline{(ij)}\ \underline{(kl)}}\}$ is the fourth-order elasticity tensor;
- $\underset{\cong}{\mathrm{M}} \in \mathbb{E}\mathrm{la}_5 := \{\underset{\cong}{\mathrm{T}} \in \overset{5}{\otimes}\mathbb{R}^2 \mid \underset{\cong}{\mathrm{T}} \in \mathbb{T}_{(ij)(kl)m}\}$ is the fifth-order elasticity tensor;
- $\underset{\approxeq}{\mathrm{A}} \in \mathbb{E}\mathrm{la}_6 := \{\underset{\approxeq}{\mathrm{T}} \in \overset{6}{\otimes}\mathbb{R}^2 \mid \underset{\approxeq}{\mathrm{T}} \in \mathbb{T}_{\underline{(ij)k}\ \underline{(lm)n}}\}$ is the sixth-order elasticity tensor.

Let's define $\mathcal{S}\mathrm{grd}$ the tensor space of the strain-gradient constitutive tensors as

$$\mathcal{S}\mathrm{grd} = \mathbb{E}\mathrm{la}_4 \oplus \mathbb{E}\mathrm{la}_5 \oplus \mathbb{E}\mathrm{la}_6. \tag{3.2}$$

A strain-gradient elastic law is defined by a triplet $\mathcal{E} := \left(\underset{\approx}{\mathrm{C}}, \underset{\cong}{\mathrm{M}}, \underset{\approxeq}{\mathrm{A}}\right) \in \mathcal{S}\mathrm{grd}$.

Concerning the symmetry classes of the tensors involved in the constitutive law

• **Classical elasticity**: the classification has been done by [20]. Their results are synthesised in the following table:

Name	Digonal	Orthotropic	Tetragonal	Isotropic
$[\mathrm{G}_{\underset{\approx}{\mathrm{C}}}]$	$[\mathrm{Z}_2]$	$[\mathrm{D}_2]$	$[\mathrm{D}_4]$	$[\mathrm{O}(2)]$
$\#_{\mathrm{indep}}(\underset{\approx}{\mathrm{C}})$	6 (5)	4	3	2

Table 3.1: The names, the sets of subgroups $[\mathrm{G}_{\underset{\approx}{\mathrm{C}}}]$ and the numbers of independent components $\#_{\mathrm{indep}}(\underset{\approx}{\mathrm{C}})$ for the 4 symmetry classes of $\mathbb{C}$. The in-parenthesis number indicates the minimal number of components of the matrix in an appropriate basis.

• **Second-order elasticity**: the classification has been done by [2], and are synthesized in the following table:

[1] In the infinitesimal setting, the strain tensor is defined from the displacement field $\underline{\mathrm{u}}$ as $\underset{\sim}{\varepsilon} := \frac{1}{2}(\underline{\mathrm{u}} \otimes \underline{\nabla} + \underline{\nabla} \otimes \underline{\mathrm{u}})$, where $\underline{\nabla}$ denotes the nabla differential operator.

Name	Digonal	Orthotropic	Tetrachiral	Tetragonal
$[\mathrm{G}_{\underset{\approx}{\mathrm{A}}}]$	$[\mathrm{Z}_2]$	$[\mathrm{D}_2]$	$[\mathrm{Z}_4]$	$[\mathrm{D}_4]$
$\#_{\mathrm{indep}}(\underset{\approx}{\mathrm{A}})$	21 (20)	12	9 (8)	6
Name	Hexachiral	Hexagonal	Hemitropic	Isotropic
$[\mathrm{G}_{\underset{\approx}{\mathrm{A}}}]$	$[\mathrm{Z}_6]$	$[\mathrm{D}_6]$	$[\mathrm{SO}(2)]$	$[\mathrm{O}(2)]$
$\#_{\mathrm{indep}}(\underset{\approx}{\mathrm{A}})$	7 (6)	5	5	4

Table 3.2: The names, the sets of subgroups $[\mathrm{G}_{\underset{\approx}{\mathrm{A}}}]$ and the numbers of independent components $\#_{\mathrm{indep}}(\underset{\approx}{\mathrm{A}})$ for the 8 symmetry classes of $\mathbb{A}$. The in-parenthesis number indicates the minimal number of components of the matrix in an appropriate basis.

2 Dimension of invariant tensor spaces in $\mathbb{R}^2$

In the the present section, we will see two different approaches to compute the number of invariant parameters for each symmetry classes. To illustrate our point, we restrict ourselves here to $Elaq$ and $Elas$, and consider only the dihedral classes. There is no difficulty in considering other situations. We will also examine the results obtained for certain groups not associated with a symmetry class. This will illustrate some interesting properties.

2.1 Bottom-up approach

The Bottom-up approach starts with the determination of the representation as a tensor product of the classical representation. Thus, we will establish here the symmetrised representations acting on $Elaq$ and $Elas$. This corresponds, respectively, to a tensor of order eight and a tensor of order twelve.

Classical elasticity: Recall that the fourth order elasticity tensor belongs to the vector space $S^2(S^2(\mathbb{R}^2))$. The representation $(\rho, \mathbb{E}\mathrm{la})$, equivalent to $(\rho, S^2(S^2(\mathbb{R}^2)))$, has the following expression

$$(\rho(\mathbf{g}))_{ijklmnop} = \frac{1}{8}\left[(g_{im}g_{jn}+g_{in}g_{jm})(g_{ko}g_{lp}+g_{kp}g_{lo})+(g_{io}g_{jp}+g_{ip}g_{jo})(g_{km}g_{ln}+g_{kn}g_{lm})\right]$$

The resulting character is

$$\chi_\rho(g) = \mathrm{tr}\rho(g) = \frac{1}{8}\left[(\mathrm{tr}g)^4 + 2(\mathrm{tr}g)^2\mathrm{tr}g^2 + 2trg^4 + 3(\mathrm{tr}g^2)^2\right]$$

Second order elasticity: The representation $(\varrho, \mathbb{E}\mathrm{la}_6)$ is equivalent to $(\varrho, S^2(S^2(\mathbb{R}^2)\otimes\mathbb{R}^2)$:

$$(\rho(\mathbf{g}))_{ijklmnoprst} = \frac{1}{8}\left[(g_{is}g_{jr}+g_{ir}g_{js})(g_{lp}g_{mo}+g_{lo}g_{mp})g_{nq}g_{kt}+(g_{ip}g_{jo}+g_{io}g_{jp})(g_{lr}g_{ms}+g_{ls}g_{mr})g_{nt}g_{kq}\right]$$

the associated character is:

$$\chi_\varrho(g) = \mathrm{tr}\varrho(g) = \frac{1}{8}\left[(\mathrm{tr}g)^6 + 2\mathrm{tr}g^2\left((\mathrm{tr}g)^4 + (\mathrm{tr}g^2)^2 + \mathrm{tr}g^4\right) + (\mathrm{tr}g)^2(\mathrm{tr}g^2)^2\right]$$

Symmetry class Z_2

The character table of the symmetry group Z_2 has been provided in Table 2.6. This group has 2 elements and 2 one-dimensional irreducible representations A and B. The following table resumes the value of the characters of the representations ρ and ϱ as well as both irreducible representations of Z_2:

Z_2	e	$\mathbf{r}_2$
χ_A	1	1
χ_B	1	-1
χ_ρ	6	6
χ_ϱ	21	21

The isotypic decomposition is calculated from:

$$m_\alpha = \frac{1}{|\mathrm{G}|} \sum_{g \in \mathrm{G}} \chi_{\rho^{(\alpha)}}(g) \chi_\rho(g).$$

The simplification $\chi_{\rho^{(\alpha)}}(g^{-1}) = \chi_{\rho^{(\alpha)}}(g)$ comes from the fact that the representation is real.

The vector spaces $\mathbb{E}\mathrm{la}_4$ and $\mathbb{E}\mathrm{la}_6$ decompose[2] as follows under the action of group Z_2:

$$\mathbb{E}\mathrm{la} = 6A \qquad \mathbb{E}\mathrm{la}_6 = 21A$$

We can see that the tensors are invariant under the action of $\mathrm{O}(2)$. This is normal because in dimension 2 any tensor of even order is Z_2 invariant.

Symmetry class $\mathbf{D_2}$ The character table of the symmetry group D_2 is presented in Table 2.6. This group has 4 elements and 4 one-dimensional irreducible representations. The following table resumes the value of the characters of the representation ρ as well as all irreducible representations of D_2:

D_2	e	$\mathbf{r}_2$	$\boldsymbol{\pi}$	$\mathbf{r}_2\boldsymbol{\pi}$
χ_{A_1}	1	1	1	1
χ_{A_2}	1	1	-1	-1
χ_{B_1}	1	-1	1	-1
χ_{B_2}	1	-1	-1	1
χ_ρ	6	6	2	2
χ_ϱ	21	21	3	3

As a consequence, thanks to the trace formula, the vector spaces $\mathbb{E}\mathrm{la}$ and $\mathbb{E}\mathrm{la}_6$ are decomposed as follows under the action of group D_2:

$$\mathbb{E}\mathrm{la}_4 = 4A_1 \oplus 2A_2 \qquad \mathbb{E}\mathrm{la}_6 = 12A_1 \oplus 9A_2$$

The invaraint coefficient correspond to the trivial representation A_1. Hence

$$\dim \mathbb{F}\mathrm{ix}(\mathbb{E}\mathrm{la}_4, \mathrm{D}_2) = 4, \qquad \dim \mathbb{F}\mathrm{ix}(\mathbb{E}\mathrm{la}_6, \mathrm{D}_2) = 12$$

in which $\mathbb{F}\mathrm{ix}(\mathbb{V}, \mathrm{G})$ indicates the dimension of the vector space of $\mathbb{V}$ elements that are G-invariant.

Symmetry class $\mathbf{D_4}$ The character table of the symmetry group D_4 is presented in Table 2.8. This group has 8 elements along with 4 one-dimensional and 1 two-dimensional irreducible representations. The following table resumes the value of the characters of the representation ρ as well as all irreducible representations of D_4:

[2] Here the abuse of notation identifying irreducible representation and their associated vector space has been made.

D_4 $\#$	e 1	$\mathbf{r}_4$ 2	$\mathbf{r}_2$ 1	$\boldsymbol{\pi}$ 2	$\mathbf{r}_4\boldsymbol{\pi}$ 2
χ_{A_1}	1	1	1	1	1
χ_{A_2}	1	1	1	-1	-1
χ_{B_1}	1	-1	1	1	-1
χ_{B_2}	1	-1	1	-1	1
χ_E	2	0	-2	0	0
χ_ρ	6	2	6	2	2
χ_ϱ	21	-3	21	3	3

As a consequence, thanks to the trace formula, the vector spaces $\mathbb{E}\mathrm{la}_4$ and $\mathbb{E}\mathrm{la}_6$ are decomposed as follows under the action of group D_4:

$$\mathbb{E}\mathrm{la}_4 = 3A_1 \oplus A_2 \oplus B_1 \oplus B_2 \qquad \mathbb{E}\mathrm{la}_6 = 6A_1 \oplus 3A_2 \oplus 6B_1 \oplus 6B_2$$

Hence

$$\dim \mathbb{F}\mathrm{ix}(\mathbb{E}\mathrm{la}_4, \mathrm{D}_4) = 3, \qquad \dim \mathbb{F}\mathrm{ix}(\mathbb{E}\mathrm{la}_6, \mathrm{D}_4) = 6$$

Finite groups $\mathbf{D_3}$ and $\mathbf{D_6}$ The character table of the symmetry groups D_3 and D_6 are presented in Table 2.7 and Table 2.10. The following table resumes the value of the characters of the representations ρ and ϱ as well as all irreducible representations of D_6. The irreducible representations can be simply restricted to the symmetry elements of D_3 noting that only irreducible representations trivial, sign and one of the two-dimensional irreducible representations are present in D_3.

D_6 $\#$	e 1	$\mathbf{r}_6$ 2	$\mathbf{r}_3$ 2	$\mathbf{r}_2$ 1	$\boldsymbol{\pi}$ 3	$\mathbf{r}_6\boldsymbol{\pi}$ 3
A_1	1	1	1	1	1	1
A_2	1	1	1	1	-1	-1
B_1	1	-1	1	-1	1	-1
B_2	1	-1	1	-1	-1	1
E_1	2	1	-1	-2	0	0
E_2	2	-1	-1	2	0	0
χ_ρ	6	0	0	6	2	2
χ_ϱ	21	0	0	21	3	3

As a consequence, the vector spaces $\mathbb{E}\mathrm{la}$ and $\mathbb{E}\mathrm{la}_6$ decompose as follows under the action of both groups D_3 and D_6:

$$\mathbb{E}\mathrm{la} = 2A_1 \oplus 2E_2 \qquad \mathbb{E}\mathrm{la}_6 = 5A_1 \oplus 2A_2 \oplus 7E_2$$

One can observe here that, for these vector spaces, both groups D_3 and D_6 have the same action.

This is a general properties of even-order tensors in $\mathbb{R}^2$. Even-order tensors cannot see the even-order symmetry $(2p+1)$, what they perceive is in fact the double-order symmetry $2(2p+1)$.

In the case of $\mathbb{E}\mathrm{la}_4$, D_6 is not a symmetry class, and the result obtained corresponds to the isotypic decomposition for $\mathrm{O}(2)$. This result can be explained by Hermann's theorem[18], if the order of the rotational symmetry is greater than the order of the tensor then the tensor is at least $\mathrm{SO}(2)$-invariant. If it also has mirror symmetry, then it is $\mathrm{O}(2)$-invariant.

Using this theorem, it is easy to determine the properties of isotropic tensors from simple calculations.

2.2 Top-down approach

The top-down approach uses the restriction operation and the branching rules to determine, from the harmonical decomposition of the tensor under the full orthogonal group $\mathrm{O}(2)$, its decomposition under the various finite subgroups of $\mathrm{O}(2)$. This branching rules are presented below:

$\mathbb{K}^0$	$\to$	A_1
$\mathbb{K}^{-1}$	$\to$	A_2
$\mathbb{K}^{2k}$	$\to$	$A_1 \oplus A_2$
$\mathbb{K}^{2k+1}$	$\to$	$B_1 \oplus B_2$

Table 3.3: Branching Rule $\mathrm{O}(2) \to \mathrm{D}_2$

$\mathbb{K}^0$	$\to$	A_1
$\mathbb{K}^{-1}$	$\to$	A_2
$\mathbb{K}^{3k}$	$\to$	$A_1 \oplus A_2$
$\mathbb{K}^{3k+1}$, $\mathbb{K}^{3k+2}$	$\to$	E

Table 3.4: Branching Rule $\mathrm{O}(2) \to \mathrm{D}_3$

$\mathbb{K}^0$	$\to$	A_1
$\mathbb{K}^{-1}$	$\to$	A_2
$\mathbb{K}^{4k}$	$\to$	$A_1 \oplus A_2$
$\mathbb{K}^{4k+1}$, $\mathbb{K}^{4k+3}$	$\to$	E
$\mathbb{K}^{4k+2}$	$\to$	$B_1 \oplus B_2$

Table 3.5: Branching Rule $\mathrm{O}(2) \to \mathrm{D}_4$

$\mathbb{K}^0$	$\to$	A_1
$\mathbb{K}^{-1}$	$\to$	A_2
$\mathbb{K}^{6k}$	$\to$	$A_1 \oplus A_2$
$\mathbb{K}^{6k+1}$, $\mathbb{K}^{6k+5}$	$\to$	E_1
$\mathbb{K}^{6k+2}$, $\mathbb{K}^{6k+4}$	$\to$	E_2
$\mathbb{K}^{6k+3}$	$\to$	$B_1 \oplus B_2$

Table 3.6: Branching Rule $\mathrm{O}(2) \to \mathrm{D}_6$

Thanks to the presented branching rules, the decomposition of the fourth-order and sixth-order elasticity tensors under the finite groups D_2, D_3, D_4 and D_6 are presented below:

G	$\mathbb{E}\mathrm{la}$	$\dim \mathrm{Fix}_{\mathbb{E}\mathrm{la}}(\mathrm{G})$
$\mathrm{O}(2)$	$2\mathbb{K}^0 \oplus \mathbb{K}^2 \oplus \mathbb{K}^4$	2
D_2	$4A_1 \oplus 2A_2$	4
D_3	$2A_1 \oplus 2E$	2
D_4	$3A_1 \oplus A_2 \oplus B_1 \oplus B_2$	3
D_6	$2A_1 \oplus 2E_2$	2

Table 3.7: Restricted G-irreducible decomposition of $\mathbb{E}\mathrm{la}$ for different finite groups

The computation of the dimension of the fixed-point space gives the minimal number of independent variables necessary to fully determine the elasticity tensor. Thanks to this computation, one can verify that, even though their isotypic decomposition differs, the fourth-order elasticity tensors with D_6, D_3 and $\mathrm{O}(2)$ symmetry all belong to the same symmetry class with a fixed-point space dimension of 2. As a consequence, a material with D_3 symmetry would appear isotropic.

G	$\mathbb{E}\mathrm{la}_6$	$\dim \mathbb{F}\mathrm{ix}_{\mathbb{E}\mathrm{la}_6}(\mathrm{G})$
$\mathrm{O}(2)$	$\mathbb{K}^{-1} \oplus 4\mathbb{K}^0 \oplus 5\mathbb{K}^2 \oplus 2\mathbb{K}^4 \oplus \mathbb{K}^6$	2
D_2	$12A_1 \oplus 9A_2$	12
D_3	$5A_1 \oplus 2A_2 \oplus 7E$	5
D_4	$6A_1 \oplus 3A_2 \oplus 6B_1 \oplus 6B_2$	6
D_6	$5A_1 \oplus 2A_2 \oplus 7E_2$	5

Table 3.8: Restricted G-irreducible decomposition of $\mathbb{E}\mathrm{la}_6$ for different finite groups

2.3 Canonical form from the decompositions

We have seen that both the top-down and the bottom-up approaches lead to the same decompositions for the various finite subgroups of $\mathrm{O}(2)$. From these decompositions, the canonical shapes of the elasticity tensors can be retrieved by choosing an appropriate basis.

to be continued...

Chapter 4 Geometry of the orbit space (en cours)

Content

- *Action*
- *Orbit and Isotropy subgroup*
- *Fixed-point space*
- *Projection operator and symmetry adapted basis*

When we make a group act on a vector space, this induces a geometry on this space. Different points in the space will have different trajectories under the action of the group of transformations, and we speak of orbits under the action of the group. These orbits are generally very diverse, some are fixed, others discrete, and most of them describe continuous sets. The nature of these orbits depends on the symmetry group, or isotropy group, of the point in space.

This setting has also numerous applications in stability and bifurcation theory. In fact, this theory originates from the problem of classifying the theoretically allowed patterns of spontaneous symmetry breaking in theories where the ground state is determined as a minimum of a G-invariant potential. This problem is transverse in physics and its formalisme range from theoretical physics (Higgs potential) to the bifurcation issues in mechanical symmetric structure. The classical references on this topic are the paper of Abud and Sartori [1], and the monography of Golubitsky et al. [19]

In the mechanics of materials, this theoretical framework makes it possible to study linear constitutive laws. In this context, we talk about the geometry of tensor spaces. The classic example is the study of the space of elasticity tensors, which is a vector space of tensors of order 4. The orbit under $\mathrm{SO}(2)$ of a so-called isotropic tensor will be reduced to a point and that of a triclinic tensor will describe a 3-variety. The classification of orbit types is a classic problem, but in mechanics we tend to talk about the determination of symmetry classes. This problem was solved in 1996 in $\mathbb{R}^3$ by Forte and Vianello [15]. More than the result as such, the major contribution of these authors was the introduction of this geometric framework.

1 Action

Consider the action of a compact group G on a finite-dimensional vector space $\mathbb{V}$ of dimension N. $\mathbb{V}$ is called the **carrier space** or the **representation space**. Recall that representations of G on $\mathbb{V}$ is a morphism from G into $\mathrm{GL}(\mathbb{V})$

$$\rho : \mathrm{G} \to \mathrm{GL}(\mathbb{V})$$

Definition 4.1 (Action)

*The **action** of $g \in \mathrm{G}$ on an element $\underline{\mathrm{v}}$ of $\mathbb{V}$ expresses as*

$$\underline{\mathrm{v}}^\star = \rho(g)\underline{\mathrm{v}}$$

♣

Example 4.1 (Tensorial action)

Let here $\mathbb{T}$ be a vector space of n-th order tensors over $\mathbb{R}^d$. The way elements of $\mathbb{T}$ are transformed with respect to $\mathrm{SO}(2)$ is as follows

Definition 4.2

$\mathrm{SO}(3)$ *acts on* $\mathbb{T}$ *through* $\star$ *defined by:*

$$\star : \mathrm{SO}(2) \times \mathbb{T} \to \mathbb{T} \quad (\mathbf{g}, \mathbf{T}) \mapsto (\mathbf{g} \star \mathbf{T})_{i_1 i_2 \dots i_n} := g_{i_1 j_1} g_{i_2 j_2} \dots g_{i_n j_n} T_{j_1 j_2 \dots j_n}$$

♣

This action is known as the **standard tensorial action**. In the mechanical literature it is sometimes also called the Rayleigh product, especially in the German school of mechanics [7]. It can be noticed that the transformation operator made from $\mathbf{g} \in \mathrm{SO}(2)$ and acting on a tensor of order n is a tensor of order $2n$.

The fact that representations could be reduced to smaller reducible up to irreducible representations transfers to carrier spaces and the carrier space associated to an irreducible representation is also called **irreducible subspace**.

Each carrier space $\mathbb{V}$ can be generated by a set of **basis vectors** $\underline{\phi}_i$, $i = 1, .., N$ where N is the dimension of the carrier space. The carrier space can then be noted: $\mathbb{V} = span(\underline{\phi}_1, .., \underline{\phi}_N)$. Note that the set of basis vectors is not unique. Additionally, two vectors in different subspaces associated with non-equivalent irreducible representations are necessarily orthogonal to each other. This is a direct consequence of the orthogonality conditions.

Theorem 4.1

Let $\underline{\phi}_i^\alpha \in \mathbb{V}^\alpha$ be a vector of the irreducible subspace $\mathbb{V}^\alpha$ and $\underline{\psi}_j^\beta \in \mathbb{V}^\beta$ be a vector of another irreducible subspace $\mathbb{V}^\beta$. Then, $< \underline{\phi}_i^\alpha | \underline{\psi}_j^\beta >= 0$. ♡

2 Orbit and Isotropy subgroups

Since we have express how G acts on $\mathbb{V}$ we can now consider the trajectory of point $\underline{v} \in \mathbb{V}$ under its action. This notion is called a G-orbit.

Definition 4.3

The orbit of (or containing) $\underline{v} \in \mathbb{V}$ is

$$\mathcal{O}(\underline{v}) = \{\underline{v}^\star = \rho(g)\underline{v} \mid g \in \mathrm{G}\}$$

♣

Orbits are subsets of $\mathbb{V}$ elements. In general different points of $\mathbb{V}$ behave differently under G: some points remain fixed, while others are transformed. It results that, usually, there not a unique orbit type on $\mathbb{V}$.

In the case G is a finite group, the orbit are discrete sets of points. When G is a continuous compact group, some orbits are manifolds, while others are discrete sets of points.

Example 4.2 Consider the action of $\mathrm{SO}(3)$ on $\mathbb{R}^3$. Consider an element $\underline{v} \in \mathbb{V}$, there are two possibilities to consider,

1. $\|\underline{v}\| \neq 0$, and $\mathrm{Orb}_{\mathrm{G}}(\underline{v})$ is a sphere of radius $\|\underline{v}\|$, ie. $\dim \mathrm{Orb}_{\mathrm{G}}(\underline{v}) = 2$.
2. $\|\underline{v}\| = 0$, and $\mathrm{Orb}_{\mathrm{G}}(\underline{v})$ is reduced to a single point, ie. $\dim \mathrm{Orb}_{\mathrm{G}}(\underline{v}) = 0$.

The case $\|\underline{v}\| \neq 0$ is said to be **generic**.

Example 4.3 Consider the action of $\mathrm{O}(2)^{(\underline{e}_3)}$ on $\mathbb{R}^3$. Consider an element $\underline{v} \in \mathbb{V}$ of coordinate (x,y,z) with respect to $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$. To ease the classification of orbits, we will consider the following functions

$$I_2(\underline{v}) = x^2 + y^2, \quad J_2(\underline{v}) = z^2$$

Concerning the orbit here are 4 possibilities to consider,

1. $(I_2(\underline{v}) > 0, J_2(\underline{v}) > 0)$, this orbit will be said to be **generic**. Its consists of two circles of radius $\sqrt{I_2}$ and located at $z = \pm\sqrt{J_2}$;
2. $(I_2(\underline{v}) = 0, J_2(\underline{v}) > 0)$, this orbit consists of two points along the axis $\underline{e}_3$ located at $z = \pm\sqrt{J_2}$;
3. $(I_2(\underline{v}) > 0, J_2(\underline{v}) = 0)$, this orbit consists of a circle of radius $\sqrt{I_2}$ located at $z = 0$;
4. $(I_2(\underline{v}) = 0, J_2(\underline{v}) = 0)$, this orbit consists of a point located at the origin.

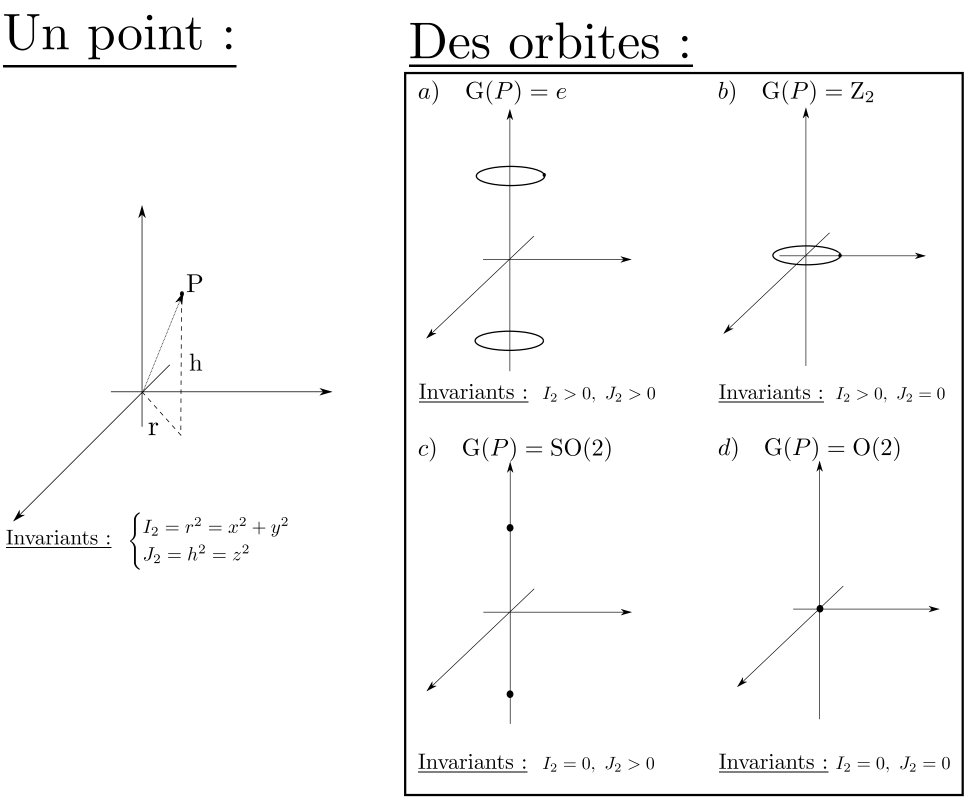


Figure 4.1: Caption

Remark In the $\mathbb{V} = \mathbb{E}\mathrm{la}$ case, the orbits of the **elastic tensors** define **elastic materials**, i.e. the elastic properties of the material independently of its specific orientation in space. This theoretically solves the question posed by Boehler et al. [9], namely whether or not two triclinic tensors measured in different laboratories correspond to the same material.

Being on the same G-orbit defines a equivalence relation among points of $\mathbb{V}$.

Definition 4.4

Two elements $\underline{v}$, $\underline{w}$ *of* $\mathbb{V}$ *are said to be G-related, denoted* $\underline{v} \sim \underline{w}$ *if* $\exists g \in \mathrm{G}$, $\underline{w} = \rho(g)\underline{v}$ *ie. if* $\underline{w} \in \mathcal{O}(\underline{v})$, *and conversely.* ♣

According to this equivalence relationship, the space $\mathbb{V}$ is partitioned into G-orbits.

Note *Since* ρ *is unitary,*

$$\forall \underline{v}^\star \in \mathcal{O}(\underline{v}), \ \|\underline{v}^\star\| = \|\underline{v}\|$$

meaning that each orbit is contained within a sphere, but generally does not coincide with it.

To understand the nature of the different orbits, the following notion, which characterises all transformations that leave a vector invariant, is central.

Definition 4.5 (Isotropy subgroup)

Let $\underline{v} \in \mathbb{V}$, *the* ***Isotropy subgroup*** *of* $\underline{v}$ *is defined as*

$$\mathrm{G}_{\underline{v}} \equiv \{g \in \mathrm{G} \mid \rho(g)\underline{v} = \underline{v}\}$$

That is, the subgroup of G *that leaves the vector* $\underline{v}$ *invariant. We say that* $\underline{v}$ *has* $\mathrm{G}_{\underline{v}}$ *symmetry.* ♣

Note *This object receives many different name according to the different communities. In mathematics it is called the* ***isotropy subgroup*** *or the* ***stabilizer****, in condensed matter physics it is the* ***little group****, and in mechanics it is called the*

symmetry group.

Note *The isotropy subgroup of G is the largest subgroup that leaves the vector $\underline{\mathrm{v}}$ invariant. As a consequence, not all subgroups $H \subset \mathrm{G}$ are isotropy subgroups.*

We have the following property

Property *Since $\rho(g)$ is linear, we have, for $\lambda \in \mathbb{R}^\star$, $\mathrm{G}_{\lambda\underline{\mathrm{v}}} = \mathrm{G}_{\underline{\mathrm{v}}}$. The isotropy subgroup is constant on a ray in $\mathbb{V}$.*

It is important to note that in the notion of the symmetry group of an object, the orientation of the symmetry elements is important. It results that when the vector is transformed, so does its symmetry group. This point is precised by the following important lemma

Lemma 4.1

The isotropy subgroups of orbits members are conjugate:

$$\mathrm{G}_{\rho(g)\underline{\mathrm{v}}} = g\mathrm{G}_{\underline{\mathrm{v}}}g^{-1}$$

♡

To speak of the symmetry of a vector intrinsically, we refer to its symmetry class, which is the conjugacy class of its symmetry group. Let us denote by $[\mathrm{G}_{\underline{\mathrm{v}}}]$ the conjugacy class of $\mathrm{G}_{\underline{\mathrm{v}}}$,

$$[\mathrm{G}_{\underline{\mathrm{v}}}] = \{g\mathrm{G}_{\underline{\mathrm{v}}}g^{-1},\ g \in \mathrm{G}\}$$

The conjugcay class is constant over each orbit.

As we just see $\underline{\mathrm{v}}$ on the same orbit have conjugate symmetry groups. It is possible to define a weaker equivalence relationship than being on the same orbit, which consists only of having a conjugate symmetry group. This weaker equivalence relation among elements of $\mathbb{V}$ is defined as follows

$$\underline{\mathrm{v}} \sim \underline{\mathrm{w}} \quad \Leftrightarrow \quad \{\exists g \in \mathrm{O}(d) | \mathrm{G}_{\underline{\mathrm{v}}} = g\mathrm{G}_{\underline{\mathrm{w}}}g^{-1}\}. \tag{4.1}$$

This relation indicates that two vectors are **equivalent** if their symmetry groups are conjugate.

The conjugacy class of a symmetry subgroup $\mathrm{G}_{\underline{\mathrm{v}}}$ of a vector $\underline{\mathrm{v}}$ is the **symmetry class** of the vector $\underline{\mathrm{v}}$, denoted $[\mathrm{G}_{\underline{\mathrm{v}}}]$. In other words, two vectors are equivalent if and only if their symmetry groups belong to the same symmetry class.

Thus, the G-orbit space can be partitioned into different sets with respect to the symmetry classes of their elements.

Now let's look at two properties relating orbits and isotropy subgroups.

Proposition 4.1

The orbit size $|\mathrm{Orb}_{\mathrm{G}} \cdot \underline{\mathrm{v}}|$ must be a factor of $|\mathrm{G}|$ and is equal to the index of $\mathrm{G}_{\underline{\mathrm{v}}}$ in G (i.e. the number of cosets of isotropy subgroups).

♠

Proof Let $\underline{\mathrm{v}} = \rho(g)\underline{\mathrm{u}}$ and $h \in \mathrm{G}_{\underline{\mathrm{v}}}$.

Then

$$\begin{aligned} & \rho(h)\underline{\mathrm{u}} = \underline{\mathrm{u}} \\ \Leftrightarrow \quad & \rho(g^{-1}\rho(g)\rho(h)\rho(g^{-1}\rho(g)\underline{\mathrm{u}} = \underline{\mathrm{u}} \\ \Leftrightarrow \quad & \rho(ghg^{-1})\rho(g)\underline{\mathrm{u}} = \rho(g)\underline{\mathrm{u}} \\ \Leftrightarrow \quad & \rho(ghg^{-1})\underline{\mathrm{v}} = \underline{\mathrm{v}} \end{aligned}$$

So if $h \in \mathrm{G}_{\underline{\mathrm{u}}}$ then $ghg^{-1} \in \mathrm{G}_{\rho(g)\underline{\mathrm{u}}}$ and $\mathrm{G}_{\rho(g)\underline{\mathrm{u}}} = g\mathrm{G}_{\underline{\mathrm{u}}}g^{-1}$ □

Note that the conjugate subgroup classification serves as a useful classification of the symmetry of an orbit and is thus sometimes referred to as the orbit-type. Thus orbits may be given a lattice structure : **the lattice of isotropy subgroups.**

Example 4.1 (Continued) Let consider the representation $(\rho^{(2)}, \mathbb{R}^2)$ of D_3.
Three example of orbits are:

Consider $\underline{u} = 0$,

$$Orb_{D_3} \cdot 0 = \{0\} \quad \text{is an orbit of size 1.}$$

Consider $\underline{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ relative to standard basis.

$$Orb_{D_3} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{bmatrix} \} \quad \text{is an orbit of size 3.}$$

Consider $\underline{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and let $\phi = \frac{1+\sqrt{3}}{2}$ and $\theta = \frac{1-\sqrt{3}}{2}$. Then

$$Orb_{D_3} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -\phi \\ -\theta \end{bmatrix}, \begin{bmatrix} -\theta \\ -\phi \end{bmatrix}, \{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -\phi \\ \theta \end{bmatrix}, \begin{bmatrix} -\theta \\ \phi \end{bmatrix} \} \quad \text{is an orbit of size 6.}$$

Three isotropy subgroups associated with these three vectors are:

$(D_3)_0 = D_3$, $(D_3)_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} = D_1^{(1)}$, $(D_3)_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} = 1$

One can observe that when the isotropy subgroup is the group itself, the only unchanged point is the origin and the orbit is of size 1. On the contrary, when the orbit has the size of the group, the isotropy subgroup is reduced to identity.

3 Fixed-point space

Definition 4.6 (Fixed-point space)

The fixed-point space of G *is defined as the vectors* $\underline{v} \in \mathbb{V}$ *which are invariant under the actions of the group* G*:*

$$\text{Fix}_{\mathbb{V}}(G) = \{\underline{v} \in \mathbb{V} | \rho(g)\underline{v} = \underline{v} \ \forall g \in G\}$$

♣

The fixed-point space corresponds to the set of all points in $\mathbb{V}$ with at least the symmetry of G. The fixed-point space dimension can be found using the trace formula:

$$\dim \text{Fix}_{\mathbb{V}}(G) = \frac{1}{|G|} \sum_{g \in G} \chi_\rho(g)$$

where χ_ρ is the character of the G-representation $(\rho, \mathbb{V})$.

Note *The fixed-point space dimension is in fact equal to multiplicity of the trivial irreducible space in the isotypic decomposition of the vector space* $\mathbb{V}$*:* $\dim \text{Fix}_{\mathbb{V}}(G) = m_1$ *with* $\mathbb{V} = \bigoplus_{\alpha=1}^{N} m_\alpha \mathbb{V}_\alpha$*. This means that* $\text{Fix}_{\mathbb{V}}(G) = \mathbb{E}_1$*, the support of the trivial isotypic component.*

Example 4.14 Over the 2D real physical space $\mathbb{R}^2$, the following fixed-point space are presented as examples: $\text{Fix}(D_3) = 0$, $\text{Fix}(\mathbf{I}) = \mathbb{R}^2$, $\text{Fix}(D_1^{(1)}) = span \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$ and $\text{Fix}(D_1^{(2)}) = span \left(\begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} \right)$.

If one considers $(\rho^{(2)}, \mathbb{R}^2, D_3)$, the representation of the action of D_3 on $\mathbb{R}^2$, then with the character table reminded below, one can see that $\dim \text{Fix}(D_3) = 0$ and $\dim \text{Fix}(D_1^{(1)}) = \dim \text{Fix}(D_1^{(2)}) = 1$.

$g \in D_3$	e	$\mathbf{r}_3$	$\mathbf{r}_3^2$	$\boldsymbol{\pi}$	$\boldsymbol{\pi}^{(2)}$	$\boldsymbol{\pi}^{(3)}$
$\chi_{\rho^{(2)}}(g)$	2	-1	-1	0	0	0

4 Projection operators and symmetry adapted basis

We introduce the projection operators onto the isotypic components of the decomposition of the vector space of any representation. Let $(\rho, \mathbb{V})$ be a representation of G and let $\rho = \bigoplus_{\alpha=1}^{N} m_\alpha \rho^{(\alpha)}$ be the decomposition of ρ into isotypic components. The support of the isotypic component $m_\alpha \rho^{(\alpha)}$, is $\mathbb{E}_\alpha = m_\alpha \mathbb{V}_\alpha$.

Theorem 4.2 (Projection operator)

For each irreducible representation $(\rho^{(\alpha)}, \mathbb{V}_\alpha)$, we set

$$\mathcal{P}_\alpha = \frac{n_\alpha}{|\mathrm{G}|} \sum_{g \in \mathrm{G}} \overline{\chi_{\rho^{(\alpha)}}}(g)\rho(g)$$

With $\chi_{\rho^{(\alpha)}}$ the character of the irreducible representation, n_α its dimension and $(\rho, \mathbb{V})$ the representation of the action of the group on the vector space $\mathbb{V}$.
Then

1. *$\mathcal{P}_\alpha$ is the projection of the vector space $\mathbb{V}$ onto $\mathbb{E}_\alpha$ under the decomposition $\mathbb{V} = \bigoplus_{\alpha=1}^{N} \mathbb{E}_\alpha$*
2. *$\mathcal{P}_\alpha \mathcal{P}_\beta = \delta_{\alpha\beta} \mathcal{P}_\alpha$*
3. *if ρ is unitary then $\mathcal{P}_\alpha$ is Hermitian.*

♡

The proof of this theorem can be found onto classical textbooks on representation theory.

Remark Note that (2) recalls that the projection operator is idempotent, meaning that $\mathcal{P}_\alpha \mathcal{P}_\alpha = \mathcal{P}_\alpha$.

Note *The projection onto the identity irreducible space $\mathbb{E}_1$ projects onto the fixed-point space of the group. Indeed if one takes a vector $\underline{v} \in \mathbb{V}$ and projects it onto the irreducible space $\mathbb{E}_1$ using the projection operator, one gets the following vector: $\underline{w} = \frac{1}{|\mathrm{G}|} \sum_{g \in \mathrm{G}} \rho(g)\underline{v}$ since all the characters of the identity irreducible representation are equal to 1. Then $\rho(h)\underline{w} = \frac{1}{|\mathrm{G}|} \sum_{g \in \mathrm{G}} \rho(h)\rho(g)\underline{v} = \frac{1}{|\mathrm{G}|} \sum_{k \in \mathrm{G}} \rho(k)\underline{v} = \underline{w}$ since ρ is a representation.*

Now that we know how to project any vector onto an irreducible space, the following problem is posed: given a G-representation $(\rho, \mathbb{V})$, how can we find a basis ψ of $\mathbb{V}$ adapted to a decomposition into irreducible representations? And let us add the restriction that ψ be chosen so that an irreducible representation which occur more than once shall appear every time in identical form. For instance:

$$g \longrightarrow \rho(g) = \begin{bmatrix} \begin{array}{c|c} \rho^{(1)}(g) & \\ \hline & \rho^{(1)}(g) \end{array} & & & \\ & \ddots & & \\ & & \rho^{(n)}(g) & \\ & & & \ddots \end{bmatrix}_\psi$$

This is more restrictive than the previous projection. To be more precise, the projection operator $\mathcal{P}_\alpha$ will project onto a space of dimension equal to that of $\mathbb{E}_\alpha = m_\alpha \mathbb{V}_\alpha$ whereas we are now trying to create a basis for this vector space so that each identical space $\mathbb{V}_\alpha$ appears identically m_α times in the decomposition. If such a basis ψ is defined, it is said to be **symmetry adapted**.

Theorem 4.3 (Symmetry adapted basis projection)

The following operator is a projection operator onto the ith symmetry adapted basis vector of $\mathbb{V}_\alpha$.

$$\mathcal{P}_{\alpha i} = \frac{n_\alpha}{|\mathrm{G}|} \sum_{g \in \mathrm{G}} \overline{\rho}_{ii}^{(\alpha)}(g)\rho(g), \quad 1 \leq i \leq n_\alpha$$

Note that there is no summation convention on the formula given here, meaning that ρ_{ii} indicates the ith component on the diagonal of the irreducible representation matrix ρ.

♡

The notion of symmetry adapted basis is often left aside by mathematical textbooks and is more often addressed in physics textbooks. It is indeed necessary in physical applications not only to project a vector onto an irreducible space but also to be able to construct its explicit decomposition on an orthonormal basis of this space. This will have application in solid mechanics to distinguish between the various deformation modes arising after an instability for instance.

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