Quiberon 2023
Approches Théoriques pour Métamatériaux

Harmonic Structure of Generalised Elasticity

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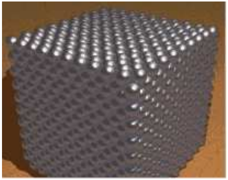
OUTLINE

ARCHITECTURED MATERIALS

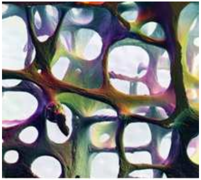
Definition

A material will be said to be architectured if:

- It presents, between its microstructure and its macrostructure, one or more other scales of organization of matter;
- If the intermediate organization scales are commensurable with those of the microstructure and/or the macrostructure.



(a) Stacking spheres



(b) Trabecular bone

(c) Coextruded steel

Characteristics of architectured materials

- Multi-functional applications and multi-physical behaviours;
- Strong anisotropy;
- Weak separation between the different scales of the material.

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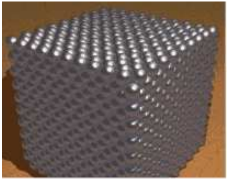
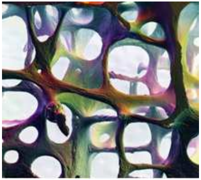


(d) Stacking spheres



(e) Trabecular bone

(f) Coextruded steel

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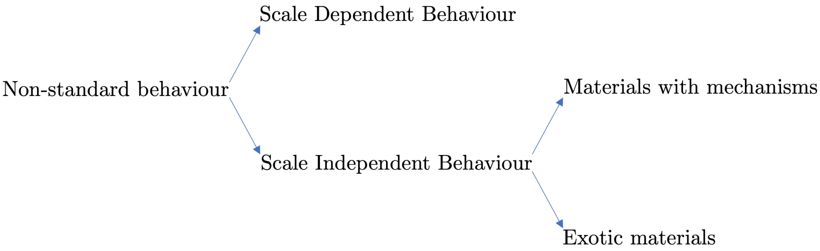
CONSEQUENCE OF THE ARCHITECTURE

A (non-contractual) typology of non-standard effects:

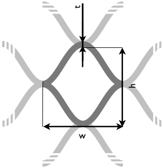


Examples of emergent behaviours
(Scale dependent behaviours)

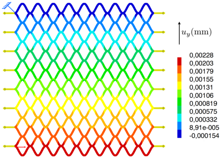
NON CENTROSYMMETRIC LATTICE

In static

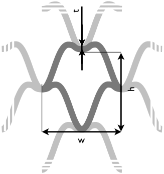
Test: Uniaxial traction on architectured material (Auffray et al. 2015; Poncelet et al. 2018)



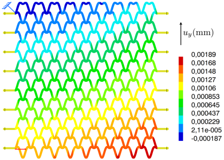
(g) Centrosymmetry



(h) Loaded



(i) Non centrosymmetry



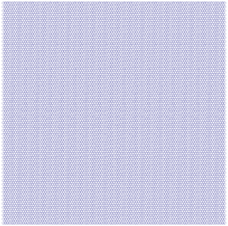
(j) Loaded

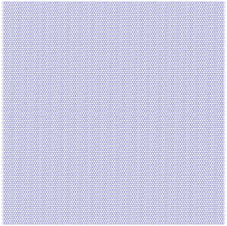
Observation: Appearance of a strain-gradient, non-standard coupling.

HEXAGONAL ANISOTROPY

Dynamic

Experiment: Propagation of elastic waves in a hexagonal lattice (Rosi et al. 2016)



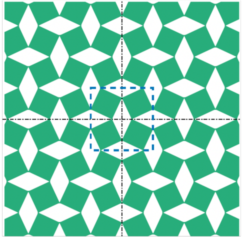


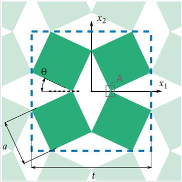
Observation: At low frequency, the propagation is isotropic, when the frequency increases the propagation becomes hexagonal.

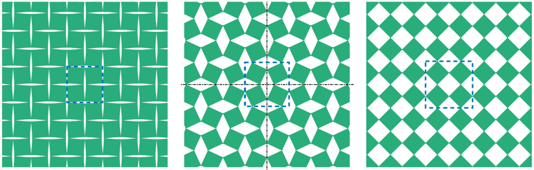
Examples of materials with mechanisms (Scale independent behaviours)

Materials with mechanisms

Auxetic snub square lattice (Durand 2022)







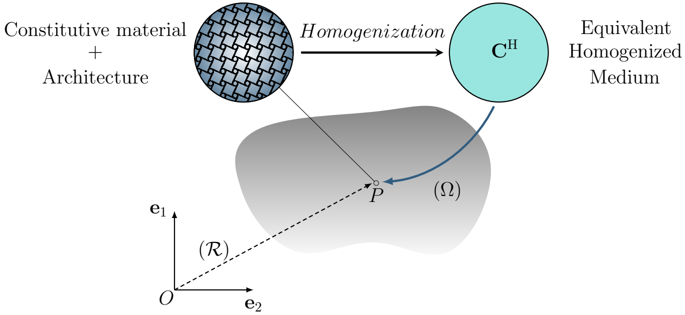
(a) (b) (c)

Observation

Isotropic strain gradient behaviour

CONTINUOUS DESCRIPTION OF STRUCTURAL EFFECTS

We want to replace the architecture ...



...to reveal its consequence at a larger scale

The effect of the mesostructure is contained in the algebraic structure of the constitutive law.

EFFECTIVE OVERALL BEHAVIOR

Some natural questions

1. What type of global continuum model should be considered?
   - Ockham's razor: the extension must be the "minimal" to capture emergent phenomena.
2. How many independent material parameters are needed to establish the model?
   - Important for identification, homogeneization, identification,...
3. What is the mechanical content of these additional parameters?
   - Important for topological optimisation, identification, model choosing,...

Assumptions

1. Small strain;
2. Linear local elasticity;
3. Theory and explicit results in 2D.

OUTLINE

COSSERAT VS. MINDLIN

In the literature, Cosserat's Elasticity (Cosserat et al. 1909) and Mindlin's SGE (Mindlin 1964) models are often opposed, but what are the ins and outs of this debate ?

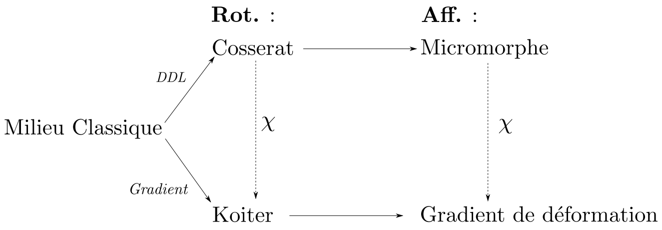
(a) F. Cosserat

(b) R.D. Mindlin

Local Generalized continua

Classical solid mechanics can be generalized (Forest 2006):

- Addition of degrees of freedom: the *micromorphic* way;
- Addition of gradients: the *strain-gradient* way;
- Combination of previous approaches.



Remark

Gradient type continua can be obtained by constraining *Micromorphic*-one.

HIGHER-ORDER THEORY: THE MICROMORPHIC FAMILY

We consider the following set of degrees of freedom:

$$\mathrm{DDL} = \{\underline{\mathrm{u}}, \underset{\sim}{\chi}\} \quad ; \quad (\underline{\mathrm{u}}, \underset{\sim}{\chi}) \in \mathbb{R}^d \times \otimes^2\mathbb{R}^d$$

The state variables associated with this kinematics are the following:

$$\mathrm{PSV} = \{\underline{\mathrm{u}} \otimes \underline{\nabla}, \underset{\sim}{\chi} \otimes \underline{\nabla}\}$$

Linear constitutive law:

$$\begin{cases} \underset{\sim}{\sigma} = \underset{\approx}{\mathrm{C}} : \underset{\sim}{\varepsilon} + \underset{\approx}{\mathrm{B}} : \underset{\sim}{\mathrm{e}} + \underset{\approx}{\mathrm{M}} \vdots \underset{\simeq}{\kappa} \\ \underset{\sim}{\mathrm{s}} = \underset{\approx}{\mathrm{B}}^T : \underset{\sim}{\varepsilon} + \underset{\approx}{\mathrm{D}} : \underset{\sim}{\mathrm{e}} + \underset{\approx}{\mathrm{E}} \vdots \underset{\simeq}{\kappa} \\ \underset{\simeq}{\tau} = \underset{\approx}{\mathrm{M}}^T : \underset{\sim}{\varepsilon} + \underset{\approx}{\mathrm{E}}^T : \underset{\sim}{\mathrm{e}} + \underset{\approx}{\mathrm{A}} \vdots \underset{\simeq}{\kappa} \end{cases}$$

with

- $\underset{\sim}{\varepsilon}$: the standard strain tensor;
- $\underset{\sim}{\mathrm{e}} = \underline{\mathrm{u}} \otimes \underline{\nabla} - \underset{\sim}{\chi}$: the relative strain tensor;
- $\underset{\simeq}{\kappa} = \underset{\sim}{\chi} \otimes \underline{\nabla}$: the micro-strain gradient.
- $\underset{\sim}{\sigma}$: the Cauchy stress tensor;
- $\underset{\sim}{\mathrm{s}}$: the relatives stress tensor;
- $\underset{\simeq}{\tau}$: the hyperstress tensor.

Choice of Kinematic Enrichment

Structure of the kinematic Enrichment (Eringen 0198; Forest et al. 2006):

$$\underset{\sim}{\chi} \in \otimes^2\mathbb{R}^3 = \underset{\sim}{\chi}^D + \underset{\sim}{\chi}^A + \underset{\sim}{\chi}^S = \underset{\sim}{\chi}^D + \underset{\simeq}{\epsilon} \cdot \underline{\phi} + \frac{1}{3}\alpha \underset{\sim}{\mathrm{I}}$$

Depending on the partial enrichments, intermediate models are obtained:

Modele	$\underset{\sim}{\chi}$	DOF
Cauchy	$\varnothing$	3
Microdilatation	α	4
Cosserat	$\underline{\phi}$	6
Microstrech	$(\underline{\phi}, \alpha)$	7
Incompressible Microstrain	$\underset{\sim}{\chi}^D$	8
Microstrain	$(\underset{\sim}{\chi}^D, \alpha)$	9
Incompressible Micromorphic	$(\underset{\sim}{\chi}^D, \underline{\phi})$	11
Micromorphic	$(\underset{\sim}{\chi}^D, \underline{\phi}, \alpha)$	12

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THE COSSERAT MODEL: MICROMORPHIC FORMULATION

We consider the following set of degrees of freedom:

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Linear constitutive law:

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with

- $\underset{\sim}{\varepsilon} = (\underline{\mathrm{u}} \otimes \underline{\nabla})^S$: the standard strain tensor;
- $\underline{\mathrm{e}} = \frac{1}{2} \underset{\simeq}{\epsilon} \cdot \underline{\omega} - \underline{\phi}$: the relative strain tensor;
- $\underset{\sim}{\kappa} = \underline{\phi} \otimes \underline{\nabla}$: the curvature tensor.
- $\underset{\sim}{\sigma} = \underset{\sim}{\sigma}^T$: the Cauchy stress tensor;
- $\underline{\mathrm{s}}$: the relatives stress tensor;
- $\underset{\sim}{\mathrm{m}}$: the couple-stress tensor.

THE COSSERAT MODEL: MICROMORPHIC FORMULATION

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with

- $\underset{\sim}{\mathrm{e}} = \underline{\mathrm{u}} \otimes \underline{\nabla} - \underset{\simeq}{\epsilon} : \underline{\phi}$: the linear stretch tensor;
- $\underset{\sim}{\kappa} = \underline{\phi} \otimes \underline{\nabla}$: the curvature tensor.
- $\underset{\sim}{\mathrm{s}}$: the asymmetric stress tensor;
- $\underset{\sim}{\mathrm{m}}$: the couple-stress tensor.

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- $\underset{\sim}{s}$: the asymmetric stress tensor;
- $\underset{\sim}{m}$: the couple-stress tensor.

LOCAL GENERALIZED CONTINUA

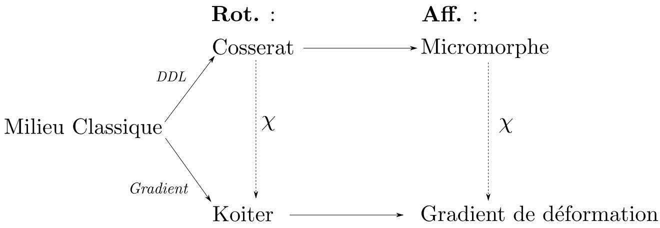
Enforcing the kinematic constraint (Bernoulli hypothesis)

$$\underset{\sim}{e} = \underline{u} \otimes \underline{\nabla} - \underset{\sim}{\chi} = 0$$

leads to

$$\underset{\sim}{\chi} = \underline{u} \otimes \underline{\nabla}, \quad \Rightarrow \quad \underset{\sim}{\chi} \otimes \underline{\nabla} = \underline{u} \otimes \underline{\nabla} \otimes \underline{\nabla}$$

meaning that micromorphic continua degenerate into strain gradient continua.



STRAIN-GRADIENT ELASTICITY

Degrees of freedom: $\mathrm{DDL} = \{\underline{u}\} \quad ; \quad \underline{u} \in \mathbb{R}^d$
State variables associated with the kinematics

$$\mathrm{PSV} = \{\underset{\sim}{\varepsilon}, \underset{\sim}{\varepsilon} \otimes \underline{\nabla}\}$$

Linear constitutive law:

$$\begin{cases} \underset{\sim}{\sigma} = \underset{\approx}{\mathrm{C}} : \underset{\sim}{\varepsilon} + \underset{\approx}{\mathrm{M}} \vdots \underset{\simeq}{\eta} \\ \underset{\simeq}{\tau} = \underset{\approx}{\mathrm{M}}^T : \underset{\sim}{\varepsilon} + \underset{\approx}{\mathrm{A}} \vdots \underset{\simeq}{\eta} \end{cases}$$

with

- $\underset{\sim}{\varepsilon}$: strain tensor;
- $\underset{\simeq}{\eta} = \underset{\sim}{\varepsilon} \otimes \underline{\nabla}$: strain gradient tensor.
- $\underset{\sim}{\sigma}$: Cauchy stress tensor;
- $\underset{\simeq}{\tau}$: hyperstress tensor.

KOITER ELASTICITY (A.K.A CONSTRAINED COUPLE STRESS ELASTICITY TOUPIN 1962)

Degrees of freedom: $\mathrm{DDL} = \{\underline{\mathrm{u}}\} \quad ; \quad \underline{\mathrm{u}} \in \mathbb{R}^d$
State variables associated with the kinematics

$$\mathrm{PSV} = \{\underset{\sim}{\varepsilon}, \underline{\omega} \otimes \underline{\nabla}\}$$

Linear constitutive law:

$$\begin{cases} \underset{\sim}{\sigma} = \underset{\approx}{\mathrm{C}} : \underset{\sim}{\varepsilon} + \underset{\approx}{\mathrm{M}} : \underset{\sim}{\kappa} \\ \underset{\sim}{\mathrm{m}} = \underset{\approx}{\mathrm{M}}^T : \underset{\sim}{\varepsilon} + \underset{\approx}{\mathrm{A}} : \underset{\sim}{\kappa} \end{cases}$$

with

- $\underset{\sim}{\varepsilon}$: strain tensor;
- $\underset{\sim}{\kappa} = \underline{\omega} \otimes \underline{\nabla}$: curvature tensor.
- $\underset{\sim}{\sigma}$: Cauchy stress tensor;
- $\underset{\sim}{\mathrm{m}}$: couple-stress tensor.

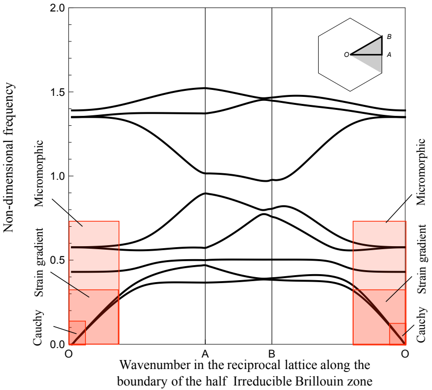
QUESTION: HOW TO CHOOSE BETWEEN STRAIN-GRADIENT AND COSSERAT MODEL

In the literature, the Cosserat and SGE models are often opposed, but

- Cosserat involves new DOFs, while SGE involves higher-gradient;
- The kinematics described by Cosserat is limited when compared to full SGE;
- The order of constitutive tensors are higher in SGE than in Cosserat;
- The dynamics feature are different (cf. tomorrow talk).

$\Rightarrow$Let us examine the ability of each model to describe higher order anisotropic effects.

ELASTODYNAMICS ASPECT



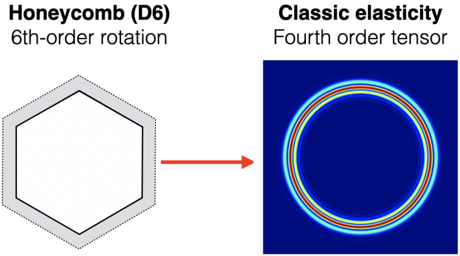
- Classical elasticity ;
  - Very longwavelength and low frequency approximation;
- Strain-Gradient elasticity;
  - Dispersion ;
- Micromorphic elasticity ;
  - Optic branches;

Strain gradient theories can not model optical branches.

HERMANN THEOREM AUFFRAY 2008; GLÜGE ET AL. 2021

Theorem

Consider $\mathcal{M}$ be a microstructure left invariant by a rotation of order n and $\mathbf{T}$ a tensor describing its effective properties. Let m be the order of the leading harmonic tensor in $\mathbf{T}$, if $n > m$ then $\mathbf{T}$ is at least $\mathrm{SO}(2)$-invariant (hemitropic).



Remark

- To describe an anisotropy of order 6, a constitutive tensor must be, at least, of 6th-order;
- To have a constitutive tensor of order 6, it is necessary to have a generalised deformation tensor of order 3.

CHOICE OF KINEMATIC ENRICHMENT (2D CASE)

Constitutive tensor depend on the gradient of $\underset{\sim}{\chi}$

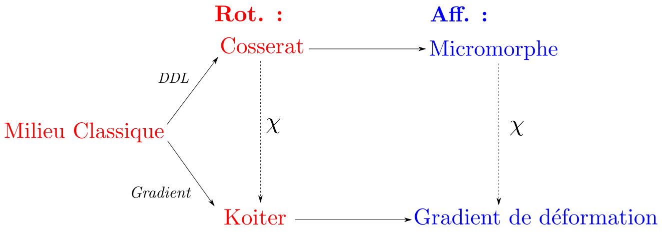
$$\underset{\sim}{\chi} \Rightarrow \underset{\simeq}{\kappa} = \underset{\sim}{\chi} \otimes \underline{\nabla}$$

To "see" an anisotropy of order 6, $\underset{\simeq}{\kappa}$ should at least be of order 3.

Modele	$\underset{\sim}{\chi}$	$\underset{\sim}{\chi} \otimes \underline{\nabla}$
Cauchy	$\varnothing$	$\varnothing$
Microdilatation	$\mathbb{K}^0$	$\mathbb{K}^1$
Cosserat	$\mathbb{K}^{-1}$	$\mathbb{K}^1$
Microstrech	$\mathbb{K}^0 \oplus \mathbb{K}^{-1}$	$2\mathbb{K}^1$
Incompressible Microstrain	$\mathbb{K}^2$	$\mathbb{K}^1 \oplus \mathbb{K}^3$
Microstrain	$\mathbb{K}^0 \oplus \mathbb{K}^2$	$2\mathbb{K}^1 \oplus \mathbb{K}^3$
Incompressible Micromorphic	$\mathbb{K}^{-1} \oplus \mathbb{K}^2$	$2\mathbb{K}^1 \oplus \mathbb{K}^3$
Micromorphic	$\mathbb{K}^0 \oplus \mathbb{K}^{-1} \oplus \mathbb{K}^2$	$3\mathbb{K}^1 \oplus \mathbb{K}^3$

Local generalized continua

Blabla



OUTLINE

Harmonic analysis of constitutive tensor spaces

Aim of the section

Introduce the basics of harmonic decomposition to analyse and compare linear constitutive models.

Outline

1. Geometrical elements;
2. Symmetry classes in $\mathbb{R}^3$;
3. Symmetry classes in $\mathbb{R}^2$;
4. Harmonic decomposition in $\mathbb{R}^2$.

The 2D setting

1. complex enough to produce non trivial results;
2. simple enough to handle explicit computations;
3. construct situations that can be extended to problems in $\mathbb{R}^3$.

THE COSSERAT MODEL: CLASSICAL FORMULATION IN $\mathbb{R}^2$

We consider the following set of degrees of freedom:

$$\mathrm{DDL} = \{\underline{\mathrm{u}}, \phi\} \quad ; \quad (\underline{\mathrm{u}}, \phi) \in \mathbb{R}^3 \times \mathbb{R}$$

The state variables associated with this kinematics are the following:

$$\mathrm{PSV} = \{\underline{\mathrm{u}} \otimes \underline{\nabla}, \underline{\nabla}\phi\}$$

Linear constitutive law:

$$\begin{cases} \underset{\sim}{\mathrm{s}} = \underset{\approx}{\mathrm{C}} : \underset{\sim}{\mathrm{e}} + \underset{\simeq}{\mathrm{K}} \cdot \underline{\kappa} \\ \underline{\mathrm{m}} = \underset{\simeq}{\mathrm{K}}^T : \underset{\sim}{\mathrm{e}} + \underset{\sim}{\mathrm{H}} \cdot \underline{\kappa} \end{cases}$$

with

- $\underset{\sim}{\mathrm{e}} = \underline{\mathrm{u}} \otimes \underline{\nabla} - \phi \underset{\sim}{\epsilon}$: the linear stretch tensor;
- $\underline{\kappa} = \underline{\nabla}\phi$: the curvature tensor.
- $\underset{\sim}{\mathrm{s}}$: the asymetric stress tensor;
- $\underline{\mathrm{m}}$: the couple-stress tensor.

Constitutive tensor space	Harmonic structure
$\underset{\approx}{\mathrm{C}} \in \mathbb{C}\mathrm{os}$	$\mathbb{K}^4 \oplus 2\mathbb{K}^2 \oplus 3\mathbb{K}^0 \oplus \mathbb{K}^{-1}$
$\underset{\simeq}{\mathrm{K}} \in \mathbb{C}\mathrm{ou}$	$\mathbb{K}^3 \oplus 3\mathbb{K}^1$
$\underset{\sim}{\mathrm{H}} \in \mathbb{R}\mathrm{ot}$	$\mathbb{K}^2 \oplus \mathbb{K}^0$

Symmetry classes (Auffray et al. 2023)

Theorem

The spaces $\mathbb{C}\mathrm{os}$, $\mathbb{C}\mathrm{ou}$ *and* $\mathbb{R}\mathrm{ot}$ *are respectively partitioned into 6, 4 and 2 symmetry classes:*

$$\begin{aligned} \mathfrak{I}(\mathbb{C}\mathrm{os}) &= \{[\mathrm{Z}_2], [\mathrm{D}_2], [\mathrm{Z}_4], [\mathrm{D}_4], [\mathrm{SO}(2)], [\mathrm{O}(2)]\}. \\ \mathfrak{I}(\mathbb{C}\mathrm{ou}) &= \{[1], [\mathrm{Z}_2^\pi], [\mathrm{D}_3], [\mathrm{O}(2)]\}. \\ \mathfrak{I}(\mathbb{R}\mathrm{ot}) &= \{[\mathrm{D}_2], [\mathrm{O}(2)]\} \end{aligned}$$

By combining these results we obtain the set of symmetry classes of the complete elasticity of Cosserat:

Theorem

The space $\mathcal{C}\mathrm{os}$ *is partitioned into 10 symmetry classes:*

$$\mathfrak{I}(\mathcal{C}\mathrm{os}) = \{[1], [\mathrm{Z}_2^\pi], [\mathrm{Z}_2], [\mathrm{D}_2], [\mathrm{Z}_3], [\mathrm{D}_3], [\mathrm{Z}_4], [\mathrm{D}_4], [\mathrm{SO}(2)], [\mathrm{O}(2)]\}.$$

Synthesis

The model is

- Sensitive to chirality and the lack of centrosymmetry;
- Cannot see anisotropy higher that 4-fold.

Symmetry classes (Auffray et al. 2023)

Theorem

The spaces $\mathbb{C}$os, $\mathbb{C}$ou *and* $\mathbb{R}$ot *are respectively partitioned into 6, 4 and 2 symmetry classes:*

$$\begin{aligned} \mathfrak{I}(\mathbb{C}\mathrm{os}) &= \{[\mathrm{Z}_2], [\mathrm{D}_2], [\mathrm{Z}_4], [\mathrm{D}_4], [\mathrm{SO}(2)], [\mathrm{O}(2)]\}. \\ \mathfrak{I}(\mathbb{C}\mathrm{ou}) &= \{[1], [\mathrm{Z}_2^\pi], [\mathrm{D}_3], [\mathrm{O}(2)]\}. \\ \mathfrak{I}(\mathbb{R}\mathrm{ot}) &= \{[\mathrm{D}_2], [\mathrm{O}(2)]\} \end{aligned}$$

By combining these results we obtain the set of symmetry classes of the complete elasticity of Cosserat:

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The global form of the constitutive law can be detailed for each symmetry class, the constitutive law has the following synthetic form:

$$\mathcal{L}_1 = \begin{pmatrix} \mathbf{A}_{\mathrm{Z}_2} & \mathbf{K}_1 \\ \mathbf{K}_1^T & \mathbf{H}_{\mathrm{D}_2} \end{pmatrix} \quad ; \quad \mathcal{L}_{\mathrm{Z}_2^\pi} = \begin{pmatrix} \mathbf{A}_{\mathrm{D}_2} & \mathbf{K}_{\mathrm{Z}_2^\pi} \\ \mathbf{K}_{\mathrm{Z}_2^\pi}^T & \mathbf{H}_{\mathrm{D}_2} \end{pmatrix} \tag{1}$$

$$\mathcal{L}_{\mathrm{Z}_2} = \begin{pmatrix} \mathbf{A}_{\mathrm{Z}_2} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{\mathrm{D}_2} \end{pmatrix} \quad ; \quad \mathcal{L}_{\mathrm{D}_2} = \begin{pmatrix} \mathbf{A}_{\mathrm{D}_2} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{\mathrm{D}_2} \end{pmatrix} \tag{2}$$

$$\mathcal{L}_{\mathrm{Z}_3} = \begin{pmatrix} \mathbf{A}_{\mathrm{SO}(2)} & \mathbf{K}_{\mathrm{D}_3} \\ \mathbf{K}_{\mathrm{D}_3}^T & \mathbf{H}_{\mathrm{O}(2)} \end{pmatrix} \quad ; \quad \mathcal{L}_{\mathrm{D}_3} = \begin{pmatrix} \mathbf{A}_{\mathrm{O}(2)} & \mathbf{K}_{\mathrm{D}_3} \\ \mathbf{K}_{\mathrm{D}_3}^T & \mathbf{H}_{\mathrm{O}(2)} \end{pmatrix} \tag{3}$$

$$\mathcal{L}_{\mathrm{Z}_4} = \begin{pmatrix} \mathbf{A}_{\mathrm{Z}_4} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{\mathrm{O}(2)} \end{pmatrix} \quad ; \quad \mathcal{L}_{\mathrm{D}_4} = \begin{pmatrix} \mathbf{A}_{\mathrm{D}_4} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{\mathrm{O}(2)} \end{pmatrix} \tag{4}$$

$$\mathcal{L}_{\mathrm{SO}(2)} = \begin{pmatrix} \mathbf{A}_{\mathrm{SO}(2)} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{\mathrm{O}(2)} \end{pmatrix} \quad ; \quad \mathcal{L}_{\mathrm{O}(2)} = \begin{pmatrix} \mathbf{A}_{\mathrm{O}(2)} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{\mathrm{O}(2)} \end{pmatrix} \tag{5}$$

Strain-gradient elasticity

Degrees of freedom: $\mathrm{DDL} = \{\underline{u}\} \quad ; \quad \underline{u} \in \mathbb{R}^d$
State variables associated with the kinematics

$$\mathrm{PSV} = \{\underset{\sim}{\varepsilon}, \underset{\sim}{\varepsilon} \otimes \underline{\nabla}\}$$

Linear constitutive law:

$$\begin{cases} \underset{\sim}{\sigma} = \underset{\approx}{C} : \underset{\sim}{\varepsilon} + \underset{\approx}{M} \,\vdots\, \underset{\simeq}{\eta} \\ \underset{\simeq}{\tau} = \underset{\approx}{M}^T : \underset{\sim}{\varepsilon} + \underset{\approxeq}{A} \,\vdots\, \underset{\simeq}{\eta} \end{cases}$$

with

- $\underset{\sim}{\varepsilon}$: strain tensor;
- $\underset{\simeq}{\eta} = \underset{\sim}{\varepsilon} \otimes \underline{\nabla}$: strain gradient tensor;.
- $\underset{\sim}{\sigma}$: Cauchy stress tensor;
- $\underset{\simeq}{\tau}$: hyperstress tensor.

Constitutive tensor space	Harmonic structure
$\underset{\approx}{C} \in \mathbb{E}\mathrm{la}$	$\mathbb{K}^4 \oplus \mathbb{K}^2 \oplus 2\mathbb{K}^0$
$\underset{\approx}{M} \in \mathbb{E}\mathrm{la}_5$	$\mathbb{K}^5 \oplus 3\mathbb{K}^3 \oplus 5\mathbb{K}^1$
$\underset{\approxeq}{A} \in \mathbb{E}\mathrm{la}_6$	$\mathbb{K}^6 \oplus 2\mathbb{K}^4 \oplus 5\mathbb{K}^2 \oplus 4\mathbb{K}^0 \oplus \mathbb{K}^{-1}$

Symmetry classes (Auffray et al. 2015)

Theorem

The spaces $\mathbb{E}\mathrm{la}$, $\mathbb{E}\mathrm{la}_5$ *and* $\mathbb{E}\mathrm{la}_6$ *are respectively partitioned into 4, 6 and 8 symmetry classes:*

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By combining these results we obtain the set of symmetry classes of the complete SGE in $\mathbb{R}^2$

Theorem

The space $\mathcal{S}\mathrm{ge}$ *is partitioned into 14 symmetry classes:*

$$\mathfrak{I}(\mathcal{S}\mathrm{ge}) = \{[1],\left[\mathrm{Z}_2^\pi\right],[\mathrm{Z}_2],[\mathrm{D}_2],[\mathrm{Z}_3],[\mathrm{D}_3],[\mathrm{Z}_4],[\mathrm{D}_4],[\mathrm{Z}_5],[\mathrm{D}_5],[\mathrm{Z}_6],[\mathrm{D}_6]\,[\mathrm{SO}(2)],[\mathrm{O}(2)]\}.$$

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\mathfrak{I}(\mathbb{E}\mathrm{la}_6) &= \{[\mathrm{Z}_2], [\mathrm{D}_2], [\mathrm{Z}_4], [\mathrm{D}_4], [\mathrm{Z}_6], [\mathrm{D}_6], [\mathrm{SO}(2)], [\mathrm{O}(2)]\}
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OUTLINE

Explicit harmonic decomposition (Auffray et al. 2021)

Harmonic structure is easy to determine, but obtaining an explicit decomposition formula is more difficult:

- the explicit decomposition is, in general, not unique;
- some explicit harmonic decompositions may lack physical interpretation;
- the complexity of the computations increases quickly with the tensor order.

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- the explicit decomposition is, in general, not unique;
- some explicit harmonic decompositions may lack physical interpretation;
- the complexity of the computations increases quickly with the tensor order.

Some methods can be found in the literature

- Spencer's Algorithm (Spencer 1970);
- Verchery's Method (Vannucci 2007; Verchery 1982);
- Zou's Approach (Zheng et al. 2000; Zou et al. 2001).

...but none of them is really satisfactory.

CRUCIAL OBSERVATION

Constitutive tensors do not come from the sky...

Figure: Case of a tenth-order tensor fallen from the sky

Crucial observation

Constitutive tensors do not come from the sky...

Figure: Case of a tenth-order tensor fallen from the sky

Main idea A constitutive tensor $\mathbf{T}$ is an element of $\mathcal{L}(\mathbb{E}, \mathbb{F})$. Let determine a decomposition of $\mathbb{T}$ compatible with those of $\mathbb{E}, \mathbb{F}$.
Main interests:

- Provide a physical content and partition of mechanical energy;
- Uniquely defined as soon as decompositions for $\mathbb{E}, \mathbb{F}$ has been chosen;
- Link with Kelvin decomposition, positive definiteness conditions are simpler.

The Clebsch-Gordan Algorithm (Auffray et al. 2021)

Definition Let $(\mathbb{E}, \mathbb{F})$ be state tensor spaces. A constitutive tensor $\mathbf{T}$ is an element of $\mathcal{L}(\mathbb{E}, \mathbb{F})$. The Clebsch-Gordan harmonic decomposition of $\mathbb{T}$ is the only harmonic decomposition of $\mathbb{T}$ compatible with those of $\mathbb{E}, \mathbb{F}$.

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Procedure:

1) State Tensor Harmonic Decomposition (STHD) Choose and compute an harmonic decomposition for elements $\underline{v} \in \mathbb{E}$ and $\underline{w} \in \mathbb{F}$.

2) Intermediate Block Decomposition (IBD) The choice of a STHD induces a decomposition of $\mathcal{L}(\mathbb{E}, \mathbb{F})$ into "blocks". This decomposition is **not irreducible**;

3) Clebsch-Gordan Harmonic Decomposition (CGHD) Each elementary block belongs to a space $\mathbb{K}^p \otimes \mathbb{K}^q$, the harmonic structure of which is known by the Clebsch-Gordan formula, and is uniquely defined.

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Interest:

- Provide a physical content and a partition of the mechanical energy;
- Uniquely defined as soon as the decompositions for $\mathbb{E}, \mathbb{F}$ has been chosen;
- Natural link with the positive definiteness conditions.

RETURNING TO STRAIN-GRADIENT ELASTICITY

Degrees of freedom: $\mathrm{DOF} = \{\underline{u}\} \quad ; \quad \underline{u} \in \mathbb{R}^d$
State variables associated with the kinematics

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Linear constitutive law (centro symmetric case):

$$\begin{cases} \underset{\sim}{\sigma} = \underset{\approx}{\mathrm{C}} : \underset{\sim}{\varepsilon} \\ \underset{\simeq}{\tau} = \underset{\approx}{\mathrm{A}} \,\vdots\, \underset{\simeq}{\eta} \end{cases}$$

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New elasticity tensor:

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Let's proceed to the decomposition of $\underset{\approx}{\mathrm{A}}$

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The first step requires the decomposition of $\mathbb{T}_{(ij)k}$

STEP 1: HARMONIC DECOMPOSITION OF $\mathbb{T}_{(ij)k}$

- The harmonic structure of $\mathbb{T}_{(ij)k}$ is:

$$\mathbb{T}_{(ij)k} \simeq \mathbb{K}^3 \oplus 2\mathbb{K}^1$$

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- Following Mindlin (Mindlin 1964), we consider the harmonic decomposition described in the following diagram:

$$\underset{\simeq}{\eta} \in \mathbb{T}_{(ij)k}$$

$$\xrightarrow{\text{Sym}} \quad \textit{Stretch-gradient}: \underset{\simeq}{\mathrm{S}} \in \mathbb{S}^3 \quad \xrightarrow{\mathcal{H}} \quad (\underset{\simeq}{\mathrm{H}}, \underline{\mathrm{v}}^s) \in (\mathbb{K}^3 \times \mathbb{K}^1_s)$$

$$\xrightarrow{\text{Id}-\text{Sym}} \quad \underset{\simeq}{\mathrm{V}}^r \in \mathbb{H}^{r(3,1)} : \text{Rotation gradient} \quad \xrightarrow{\mathcal{H}} \quad \underline{\mathrm{v}}^r \in \mathbb{K}^1_r$$

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$\xrightarrow{\text{Id}-\text{Sym}}$ $\underset{\simeq}{\mathrm{V}}^r \in \mathbb{H}^{r(3,1)}$: Rotation gradient $\xrightarrow{\mathcal{H}}$ $\underline{\mathrm{v}}^r \in \mathbb{K}^1_r$

- An associated set of orthogonal projectors $(\underset{\approxeq}{\mathrm{P}}^3, \underset{\approxeq}{\mathrm{P}}^{1s}, \underset{\approxeq}{\mathrm{P}}^{1r})$ is defined.

Step 2 & 3: Block Decomposition of $\mathbb{E}\mathrm{la}_6$

From the harmonic decomposition of $\mathbb{T}_{(ij)k}$, the relation:

$$\underset{\simeq}{\tau} = \underset{\approx}{\mathrm{A}} \,\vdots\, \underset{\simeq}{\eta}$$

can be expanded and block-decomposed:

$$\begin{pmatrix} \underset{\simeq}{\tau}^S \\ \underset{\simeq}{\tau}^R \end{pmatrix} = \begin{pmatrix} \underset{\approx}{\mathrm{A}}^{SS} & \underset{\approx}{\mathrm{A}}^{RS} \\ \underset{\approx}{\mathrm{A}}^{SR} & \underset{\approx}{\mathrm{A}}^{RR} \end{pmatrix} \begin{pmatrix} \underset{\simeq}{\eta}^S \\ \underset{\simeq}{\eta}^R \end{pmatrix} \Rightarrow \begin{pmatrix} \underset{\simeq}{\tau}^{S,3} \\ \underset{\simeq}{\tau}^{S,1} \\ \underset{\simeq}{\tau}^{R,1} \end{pmatrix} = \begin{pmatrix} \underset{\approx}{\mathrm{A}}^{SS,33} & \underset{\approx}{\mathrm{A}}^{SS,31} & \underset{\approx}{\mathrm{A}}^{SR,31} \\ \underset{\approx}{\mathrm{A}}^{SS,13} & \underset{\approx}{\mathrm{A}}^{SS,11} & \underset{\approx}{\mathrm{A}}^{SR,11} \\ \underset{\approx}{\mathrm{A}}^{RS,13} & \underset{\approx}{\mathrm{A}}^{RS,11} & \underset{\approx}{\mathrm{A}}^{RR,11} \end{pmatrix} \begin{pmatrix} \underset{\simeq}{\eta}^{S,3} \\ \underset{\simeq}{\eta}^{S,1} \\ \underset{\simeq}{\eta}^{R,1} \end{pmatrix}$$

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with

$$\begin{pmatrix} \underset{\approx}{\mathrm{A}}^{SS,33} & \underset{\approx}{\mathrm{A}}^{SS,31} & \underset{\approx}{\mathrm{A}}^{SR,31} \\ & \underset{\approx}{\mathrm{A}}^{SS,11} & \underset{\approx}{\mathrm{A}}^{SR,11} \\ & & \underset{\approx}{\mathrm{A}}^{RR,11} \end{pmatrix} \in \begin{pmatrix} \mathbb{K}^6 \oplus \mathbb{K}^0 & \mathbb{K}^4 \oplus \mathbb{K}^2 & \mathbb{K}^4 \oplus \mathbb{K}^2 \\ & \mathbb{K}^2 \oplus \mathbb{K}^0 & \mathbb{K}^2 \oplus \mathbb{K}^0 \oplus \mathbb{K}^{-1} \\ & & \mathbb{K}^2 \oplus \mathbb{K}^0 \end{pmatrix}$$

- Stretch-gradient stiffnesses;
- Rotation-gradient stiffnesses;
- Coupling stiffnesses.

THE EXPLICIT CLEBSCH-GORDAN HARMONIC DECOMPOSITION

Proposition

The tensor $\underset{\approx}{\mathrm{A}} \in \mathbb{E}\mathrm{la}_6$ *admits the uniquely defined Clebsch-Gordan Harmonic decomposition associated to the family of projectors* $(\underset{\approx}{\mathrm{P}}^{3}, \underset{\approx}{\mathrm{P}}^{1s}, \underset{\approx}{\mathrm{P}}^{1r})$

$$\begin{aligned}
\underset{\approx}{\mathrm{A}} \quad = \quad & \underset{\approx}{\mathrm{H}}^{(6)} + \frac{4}{3}(\underset{\approx}{\mathrm{H}}^{(4,s)} \cdot \underset{\approx}{\phi}^{s(3,1)} + \underset{\approx}{\phi}^{s(3,1)} \cdot \underset{\approx}{\mathrm{H}}^{(4,s)}) + \frac{3}{2}(\underset{\approx}{\mathrm{H}}^{(4,r)} \cdot \underset{\approx}{\phi}^{r(1,3)} + \underset{\approx}{\phi}^{r(3,1)} \cdot \underset{\approx}{\mathrm{H}}^{(4,r)}) \\
& + \frac{16}{9}\underset{\approx}{\phi}^{s(3,1)} \cdot \underset{\sim}{\mathrm{h}}^{1s,1s} \cdot \underset{\approx}{\phi}^{s(3,1)} + \frac{9}{4}\underset{\approx}{\phi}^{r(3,1)} \cdot \underset{\sim}{\mathrm{h}}^{1r,1r} \cdot \underset{\approx}{\phi}^{r(1,3)} \\
& + \frac{4}{3}(\underset{\approx}{\phi}^{(4,2)} : \underset{\sim}{\mathrm{h}}^{3,1s} \cdot \underset{\approx}{\phi}^{s(3,1)} + \underset{\approx}{\phi}^{s(3,1)} \cdot \underset{\sim}{\mathrm{h}}^{3,1s} : \underset{\approx}{\phi}^{2,4}) \\
& + \frac{3}{2}(\underset{\approx}{\phi}^{(4,2)} : \underset{\sim}{\mathrm{h}}^{3,1r} \cdot \underset{\approx}{\phi}^{r(1,3)} + \underset{\approx}{\phi}^{r(3,1)} \cdot \underset{\sim}{\mathrm{h}}^{3,1r} : \underset{\approx}{\phi}^{2,4}) \\
& + 2\left(\underset{\approx}{\phi}^{s(3,1)} \cdot \underset{\sim}{\mathrm{h}}^{1s,1r} \cdot \underset{\approx}{\phi}^{r(1,3)} + \underset{\approx}{\phi}^{r(3,1)} \cdot \underset{\sim}{\mathrm{h}}^{1s,1r} \cdot \underset{\approx}{\phi}^{s(3,1)}\right) \\
& + \frac{\alpha^{3,3}}{2}\underset{\approx}{\mathrm{P}}^{3} + \frac{2}{3}\underset{\sim}{\alpha}^{1s,1s}\underset{\approx}{\mathrm{P}}^{1s} + \frac{3}{4}\alpha^{1r,1r}\underset{\approx}{\mathrm{P}}^{1r} + \alpha^{1s,1r}\left(\underset{\approx}{\phi}^{s(3,1)} \cdot \underset{\approx}{\phi}^{r(1,3)} + \underset{\approx}{\phi}^{r(3,1)} \cdot \underset{\approx}{\phi}^{s(3,1)}\right) \\
& + \beta^{1s,1r}\left(\underset{\approx}{\phi}^{s(3,1)} \cdot \underset{\sim}{\epsilon} \cdot \underset{\approx}{\phi}^{r(1,3)} - \underset{\approx}{\phi}^{r(3,1)} \cdot \underset{\sim}{\epsilon} \cdot \underset{\approx}{\phi}^{s(3,1)}\right)
\end{aligned}$$

in which
$\underset{\approx}{\mathrm{H}}^{(6)}, \underset{\approx}{\mathrm{H}}^{(4,s)}, \underset{\approx}{\mathrm{H}}^{(4,r)}, \underset{\sim}{\mathrm{h}}^{3,1s}, \underset{\sim}{\mathrm{h}}^{3,1r}, \underset{\sim}{\mathrm{h}}^{1s,1r}, \underset{\sim}{\mathrm{h}}^{1s,1s}, \underset{\sim}{\mathrm{h}}^{1r,1r}, \alpha^{3,3}, \alpha^{1s,1s}, \alpha^{1r,1r}, \alpha^{1r,1s}, \beta^{1r,1s}$
are elements of $\mathbb{K}^6 \times (\mathbb{K}^4)^2 \times (\mathbb{K}^2)^5 \times (\mathbb{R})^5$.

Bouquet of harmonic vectors: Generic Case

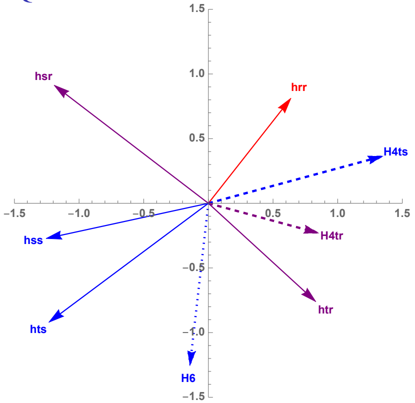


Figure: Harmonic bouquet of $\underset{\approx}{\mathrm{A}}$

Squared norms (degree 2)

Invariant	Expression
I_2	$\underset{\approx}{\mathrm{H}}^{(6)} \,\text{:::}\, \underset{\approx}{\mathrm{H}}^{(6)}$
J_2	$\underset{\approx}{\mathrm{H}}^{(4,s)} :: \underset{\approx}{\mathrm{H}}^{(4,s)}$
K_2	$\underset{\approx}{\mathrm{H}}^{(4,r)} :: \underset{\approx}{\mathrm{H}}^{(4,r)}$
L_2	$\underset{\sim}{\mathrm{h}}^{3,1s} : \underset{\sim}{\mathrm{h}}^{3,1s}$
M_2	$\underset{\sim}{\mathrm{h}}^{1s,1s} : \underset{\sim}{\mathrm{h}}^{1s,1s}$
N_2	$\underset{\sim}{\mathrm{h}}^{1r,1r} : \underset{\sim}{\mathrm{h}}^{1r,1r}$
O_2	$\underset{\sim}{\mathrm{h}}^{3,1r} : \underset{\sim}{\mathrm{h}}^{3,1r}$
P_2	$\underset{\sim}{\mathrm{h}}^{1s,1r} : \underset{\sim}{\mathrm{h}}^{1s,1r}$

Hemitropic Moduli (degree 1)

$\alpha^{3,3}$	$\alpha^{1s,1s}$	$\alpha^{1r,1r}$	$\alpha^{1s,1r}$	$\beta^{1s,1r}$

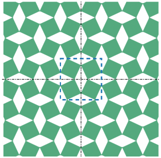
Vanishing of the squared norms

The vanishing of the harmonic norms are sufficient to parametrize numerous situations

OUTLINE

Example 1: Auxetic square (Durand 2022)

$$\begin{pmatrix} \underset{\simeq}{\tau}^{d,3} \\ \underset{\simeq}{\tau}^{d,1} \\ \underset{\simeq}{\tau}^{h,1} \end{pmatrix} = \begin{pmatrix} \mathbb{K}^0 & \mathbb{K}^4 & \mathbb{K}^4 \\ & \mathbb{K}^0 & \mathbb{K}^0 \\ & & \mathbb{K}^0 \end{pmatrix} \begin{pmatrix} \underset{\simeq}{\eta}^{d,3} \\ \underset{\simeq}{\eta}^{d,1} \\ \underset{\simeq}{\eta}^{h,1} \end{pmatrix}$$



Proposition

The tensor $\underset{\approx}{\mathrm{A}}^{\mathrm{D}_4} \in \mathbb{E}\mathrm{la}_6$ *admits the Clebsch-Gordan Harmonic decomposition*

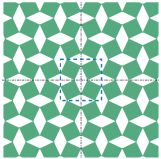
$$\begin{aligned} \underset{\approx}{\mathrm{A}} &= \left(\underset{\approx}{\mathrm{H}}^{3,1d} \cdot \underset{\approx}{\phi}^{d\{1,3\}} + \underset{\approx}{\phi}^{d\{3,1\}} \cdot \underset{\approx}{\mathrm{H}}^{3,1d}\right) + 2\left(\underset{\approx}{\mathrm{H}}^{3,1h} \cdot \underset{\approx}{\phi}^{h\{1,3\}} + \underset{\approx}{\phi}^{h\{3,1\}} \cdot \underset{\approx}{\mathrm{H}}^{3,1h}\right) \\ &+ \frac{\alpha^{3,3}}{2} \underset{\approx}{\mathrm{P}}^{(3,3)} + \frac{1}{2}\alpha^{1d,1d} \underset{\approx}{\mathrm{P}}^{(3,1d)} \\ &+ \alpha^{1h,1h} \underset{\approx}{\mathrm{P}}^{(3,1h)} + \alpha^{1d,1h}\left(\underset{\approx}{\phi}^{d\{3,1\}} \cdot \underset{\approx}{\phi}^{h\{1,3\}} + \underset{\approx}{\phi}^{h\{3,1\}} \cdot \underset{\approx}{\phi}^{d\{1,3\}}\right) \end{aligned}$$

Remark

The spheric strain is a soft mode, only its gradient play a role for the mechanical energy

EXAMPLE 1: AUXETIC SQUARE (DURAND 2022)

$$\begin{pmatrix} \underset{\simeq}{\tau}^{d,3} \\ \underset{\simeq}{\tau}^{d,1} \\ \underset{\simeq}{\tau}^{h,1} \end{pmatrix} = \begin{pmatrix} \mathbb{K}^0 & \mathbb{K}^4 & \mathbb{K}^4 \\ & \mathbb{K}^0 & \mathbb{K}^0 \\ & & \mathbb{K}^0 \end{pmatrix} \begin{pmatrix} \underset{\simeq}{\eta}^{d,3} \\ \underset{\simeq}{\eta}^{d,1} \\ \underset{\simeq}{\eta}^{h,1} \end{pmatrix}$$



Proposition

For the considered soft-mode mechanism, the projected tensor $\underset{\approx}{\mathrm{A}}^{\star}$ *reduces to*

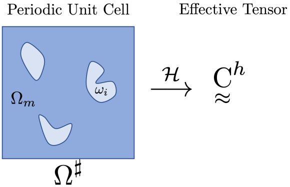
$$\underset{\approx}{\mathrm{A}}^{\star} = \alpha^{1h,1h} \underset{\approx}{\mathrm{P}}^{(3,1h)}$$

Conclusion

Auxetic square possesses an isotropic strain gradient behaviour

EXAMPLE 2: SECOND ORDER INVERSE CELL PROBLEM (**Cal21**)

- **Inverse cell problem**: Topology optimisation of a periodic unit cell with cost function defined on the effective elasticity tensor



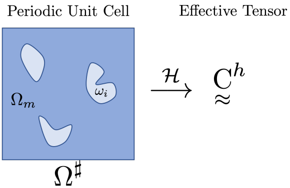
Cost function:

$$J(\Omega_m) = f(\underset{\approx}{\mathbb{C}}^h) + \lambda|\Omega_m|$$

- The classical algorithm is based on the topological derivative of $\underset{\approx}{\mathbb{C}}^h$ (Amstutz et al. 2010);

Example 2: second order inverse cell problem (Cal21)

• **Inverse cell problem**: Topology optimisation of a periodic unit cell with cost function defined on the effective elasticity tensor



Cost function:

$$J(\Omega_m) = f(\underset{\approx}{\mathrm{C}}^h) + \lambda|\Omega_m|$$

• The classical algorithm is based on the topological derivative of $\underset{\approx}{\mathrm{C}}^h$ (Amstutz et al. 2010);

Recent extensions

- Topological derivative of homogenised tensors of order 5 and 6 (Calisti et al. 2021);
- Extension of the S. Amstutz method to strain-gradient;
- Functionals expressed from tensor invariants.

EXAMPLE 2: SECOND ORDER INVERSE CELL PROBLEM (Cal21)

Goal : Design of a tetrachiral periodic material.

Harmonic structure of $\underset{\approx}{\mathrm{A}}$**:**

$$\begin{pmatrix} \mathbb{K}^0 & \mathbb{K}^4 & \mathbb{K}^4 \\ & \mathbb{K}^0 & \mathbb{K}^0 \oplus \mathbb{K}^{-1} \\ & & \mathbb{K}^0 \end{pmatrix}$$

Minimisation of $\beta^{1r,1s} \in \mathbb{K}^{-1}$, in components:

$$\beta^{1r,1s} \in \mathbb{K}^{-1} = \frac{1}{2}\left(A_{111112} - A_{111121} + A_{122112} + A_{122222} - A_{221121} - A_{221222}\right);$$

EXAMPLE 2: SECOND ORDER INVERSE CELL PROBLEM (**Cal21**)

Goal : Design of a tetrachiral periodic material.
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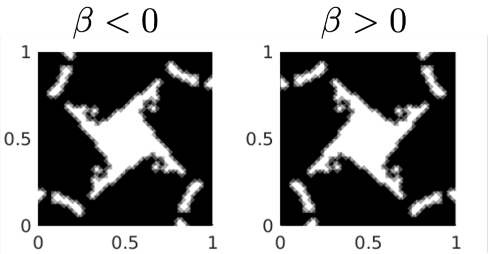
Resulting designs



Bouquet of harmonic vectors: D_6-invariance (hexatrope)

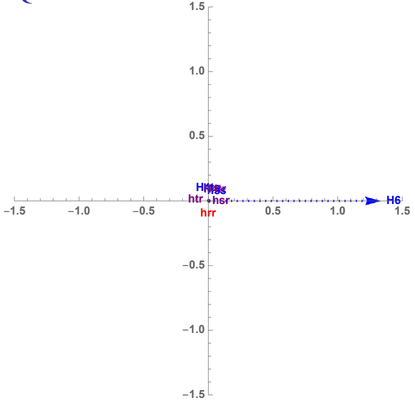


Figure: Harmonic bouquet of $\underset{\approx}{\mathrm{A}}$

Squared norms (degree 2)

Invariant	Expression
I_2	$\underset{\approx}{\mathrm{H}}^{(6)} ::: \underset{\approx}{\mathrm{H}}^{(6)}$
J_2	$\underset{\approx}{\mathrm{H}}^{(4,s)} :: \underset{\approx}{\mathrm{H}}^{(4,s)}$
K_2	$\underset{\approx}{\mathrm{H}}^{(4,r)} :: \underset{\approx}{\mathrm{H}}^{(4,r)}$
L_2	$\underset{\sim}{\mathrm{h}}^{3,1s} : \underset{\sim}{\mathrm{h}}^{3,1s}$
M_2	$\underset{\sim}{\mathrm{h}}^{1s,1s} : \underset{\sim}{\mathrm{h}}^{1s,1s}$
N_2	$\underset{\sim}{\mathrm{h}}^{1r,1r} : \underset{\sim}{\mathrm{h}}^{1r,1r}$
O_2	$\underset{\sim}{\mathrm{h}}^{3,1r} : \underset{\sim}{\mathrm{h}}^{3,1r}$
P_2	$\underset{\sim}{\mathrm{h}}^{1s,1r} : \underset{\sim}{\mathrm{h}}^{1s,1r}$

Hemitropic Moduli (degree 1)

$\alpha^{3,3}$	$\alpha^{1s,1s}$	$\alpha^{1r,1r}$	$\alpha^{1s,1r}$	$\beta^{1s,1r}$

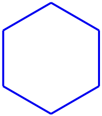
Vanishing of the squared norms

$$\underset{\approx}{\mathrm{T}} \in \overline{\Sigma}_{[\mathrm{D}_6]} \text{ iff } J_2 + K_2 + L_2 + M_2 + N_2 + O_2 + P_2 + (\beta^{1s,1r})^2 = 0$$

EXAMPLE 3: HEXATROPIC WAVE PROPAGATION (D_6)

$$\begin{pmatrix} \underset{\sim}{\tau}^{S,3} \\ \underset{\sim}{\tau}^{S,1} \\ \underset{\sim}{\tau}^{R,1} \end{pmatrix} = \begin{pmatrix} \mathbb{K}^6 \oplus \mathbb{K}^0 & 0 & 0 \\ & \mathbb{K}^0 & \mathbb{K}^0 \\ & & \mathbb{K}^0 \end{pmatrix} \begin{pmatrix} \underset{\sim}{\eta}^{S,3} \\ \underset{\sim}{\eta}^{S,1} \\ \underset{\sim}{\eta}^{R,1} \end{pmatrix}$$

- Stretch-gradient;
- Rotation-gradient;
- Coupling.



Proposition

The tensor $\underset{\approx}{\mathrm{A}}^{D_6} \in \mathbb{E}\mathrm{la}_6$ *admits Clebsch-Gordan Harmonic decomposition*

$$\underset{\approx}{\mathrm{A}}^{D_6} = \underset{\approx}{\mathrm{H}}^{(6)} + \frac{\alpha^{3,3}}{2}\underset{\approx}{\mathrm{P}}^{3} + \frac{2}{3}\alpha^{1s,1s}\underset{\approx}{\mathrm{P}}^{1s} + \frac{3}{4}\alpha^{1r,1r}\underset{\approx}{\mathrm{P}}^{1r} + \alpha^{1s,1r}\left(\underset{\approx}{\phi}^{s(3,1)} \cdot \underset{\approx}{\phi}^{r(1,3)} + \underset{\approx}{\phi}^{r(3,1)} \cdot \underset{\approx}{\phi}^{s(1,3)}\right)$$

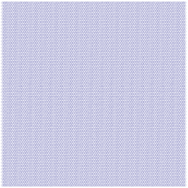
in which $\underset{\approx}{\mathrm{H}}^{(6)}, \alpha^{3,3}, \alpha^{1s,1s}, \alpha^{1r,1r}, \alpha^{1r,1s}$ *are elements of* $\mathbb{K}^6 \times (\mathbb{R})^4$.

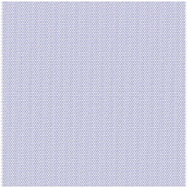
Remark

The anisotropy in the D_6 case only concerns the stretch gradient stiffness.

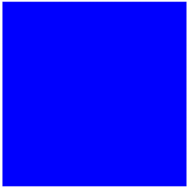
Example 3: hexatropic wave propagation (D_6) (Rosi et al. 2016)

With explicit microstructure:

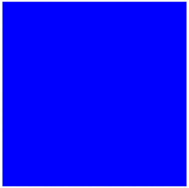




Once homogenized:



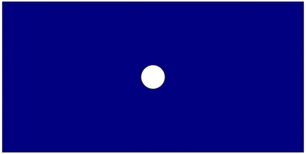
(c) Low frequency



(d) High frequency

PSEUDO CLOAKING EFFECT IN ARCHITECTURED MATERIALS ROSI ET AL. 2019

Propagation within a strain-gradient continuum

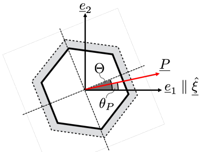


Pseudo cloaking effect in architectured materials (Rosi et al. 2019)

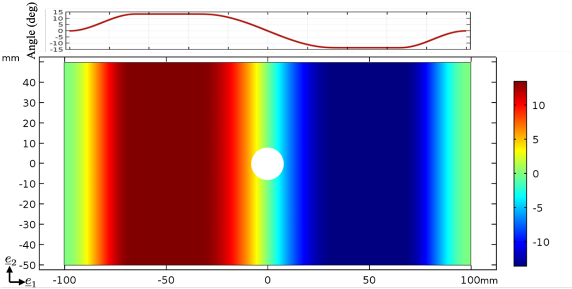
In condensed form

$$\underset{\approx}{\mathrm{A}}^{\mathrm{D}_6}(\Theta) = \underset{\approx}{\mathrm{A}}^{\mathrm{O}(2)} + \underbrace{a_D \underset{\approx}{\mathrm{A}}(\Theta)}_{\underset{\approx}{\mathrm{H}}^{(6)}}$$

Schematic representation of the angles involved:



Distribution of the material orientation angle $\Theta_{opt}(x_1)$ within a sample



PSEUDO CLOAKING EFFECT IN ARCHITECTURED MATERIALS ROSI ET AL. 2019

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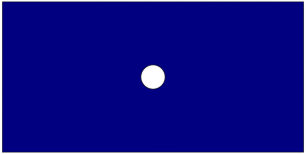
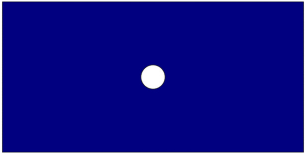


Pseudo cloaking effect in architectured materials Rosi et al. 2019

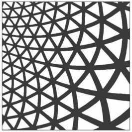
Propagation within a strain-gradient continuum



On-going work:

De-Homogenization of the effective structure:

- conformal transformation;
- …



EXAMPLE 4: INVERSE PROBLEM

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